THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, October 25, 1902. About forty-five persons attended the morning and afternoon sessions, including the following thirty-five members of the Society:

Professor Maxime Bôcher, Professor Joseph Bowden, Dr. J. E. Clarke, Professor F. N. Cole, Dr. W. S. Dennett, Professor A. M. Ely, Dr. William Findlay, Professor T. S. Fiske, Miss Ida Griffiths, Dr. E. R. Hedrick, Dr. G. W. Hill, Dr. A. A. Himowich, Dr. E. V. Huntington, Dr. S. A. Joffe, Dr. Edward Kasner, Dr. G. H. Ling, Mr. H. B. Mitchell, Dr. I. E. Rabinovitch, Professor J. K. Rees, Mr. C. H. Rockwell, Dr. P. L. Saurel, Miss I. M. Schottenfels, Professor C. A. Scott, Professor D. E. Smith, Dr. Virgil Snyder, Dr. H. F. Stecker, Miss Mary Underhill, Professor E. B. Van Vleck, Professor L. A. Wait, Mr. H. E. Webb, Professor J. B. Webb, Professor A. G. Webster, Miss E. C. Williams, Dr. R. G. Wood, Professor R. S. Woodward.

Vice-President Professor Maxime Bôcher presided at the morning session, Ex-President Professor R. S. Woodward at the afternoon session. The Council announced the election of the following persons to membership in the Society: Professor Sir R. S. Ball, Cambridge University, England; Dr. Otto Dunkel, Wesleyan University, Middletown, Conn.; Mr. W. H. Osborne, Purdue University, Lafayette, Ind.; Professor H. L. Rietz, Butler College, Indianapolis, Ind.; Professor G. H. Scott, Yankton College, Yankton, So. Dak.; Professor J. N. Van der Vries, University of Kansas, Lawrence, Kan.; Professor B. F. Yanney, Mount Union College, Alliance, Ohio; Mr. W. H. Young, Cambridge University, England. Seven applications for membership in the Society were received.

A list of nominations to be placed on the official ballot for the annual election of officers and other members of the Council was reported. A committee consisting of the Secretary and Professors W. F. Osgood, Oskar Bolza, James Pierpont, and

G. A. Miller was appointed to make arrangements for the next summer meeting and colloquium of the Society.

The following papers were presented at this meeting:

(1) DR. E. R. HEDRICK: "On the foundations of mechanics" (preliminary communication).

(2) DR. E. V. HUNTINGTON: "Definitions of a commutative group by independent postulates."

(3) PROFESSOR PETER FIELD: "On the infinite branches of plane curves which have no point singularities."

(4) DR. EDWARD KASNER: "The apolarity of double binary forms."

(5) PROFESSOR MAXIME BÔCHER: "An application of the Riemann-Darboux generalization of Green's theorem."

(6) PROFESSOR MAXIME BÔCHER: "Note on Laplace's equation."

(7) DR. VIRGIL SNYDER: "On the quintic scrolls having three double conics."

(8) MISS I. M. SCHOTTENFELS: "Note on the types of groups of order p^n every element of which, except identity, is of order p" (preliminary communication).

(9) DR. L. P. EISENHART: "Surfaces referred to their lines of length zero."

(10) PROFESSOR L. E. DICKSON: "Three sets of generational relations defining the abstract simple group of order 504."

(11) PROFESSOR L. E. DICKSON: "Generational relations defining the abstract simple group of order 660."

(12) DR. G. H. LING: "The approximate representation of functions by means of functions defined by quadratic equations."

(13) DR. C. N. HASKINS: "On the invariants of differential forms of degree higher than two."

Dr. Haskins's paper was presented to the Society through Professor Bôcher. In the absence of the authors, Professor Field's paper was read by Dr. Snyder, that of Dr. Haskins by Professor Bôcher, and the papers of Dr. Eisenhart and Professor Dickson were read by title.

The papers of Dr. Snyder, Dr. Eisenhart and Professor Dickson will appear in the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In Dr. Hedrick's preliminary report, an attempt is made to establish a logical system of axioms, sufficient for the deriva-

tion of the elementary theorems in the mechanics of a system of particles. The ideas of space and time are made fundamental, space being taken as that configuration of points, lines and planes defined by Hilbert's set of axioms in his Grundlagen der Geometrie, while time is defined as a parameter, in terms of which the equations of any curve in space may so be expressed that these equations represent the motion of the particle in the ordinary sense. Axioms are then laid down which fix (or define) the ideas "before," "after," etc.

Mass is now defined in terms of time and space, on the basis of Newton's law of universal gravitation, previous axioms having required a certain form of motion in the case when two particles are isolated in space. Force is then defined as usual, and the remaining axioms necessary to the development of the elementary theorems are laid down. This preliminary report is intended to be a forerunner of a more complete discussion, and the purpose of presenting it at this time is to invite suggestion and criticism.

2. The definitions of an abelian (commutative) group given in Dr. Huntington's paper are simple modifications of the author's definitions of a general group published in the last volume of the BULLETIN.* According to the first definition, a set of elements, with a rule of combination $\circ$, will be an abelian group when the following three postulates are satisfied: 1° $a \circ b = b \circ a$, whenever a, b and $b \circ a$ belong to the set. 2° $(a \circ b) \circ c = a \circ (b \circ c)$, whenever a, b, $a \circ b$, $b \circ c$ and $a \circ (b \circ c)$ belong to the set. 3° For every two elements a and b there is an element x in the set, such that $a \circ x = b$. The second definition involves four postulates. The independence of the postulates of each set is established both for finite and for infinite groups. The paper will appear in the *Transactions*.

3. Professor Field's paper considers the form of sextic curves which have no point singularities and which have two essential infinite branches. It is shown that such curves may be unipartite or multipartite. The form and equation of a multipartite curve of degree $3n$ with no point singularities and with n infinite branches is thus obtained, and also the form and equation of a non-singular unipartite curve of degree $12n$ with

* Cf., on this subject an article by Professor E. H. Moore in the *Transactions* for October, 1902.

$4n$ essential infinite branches. The question as to the maximum number of essential infinite branches which a curve without point singularities may possess is then considered, it being shown that such a curve may have $n-4$ essential infinite branches and that this is the maximum number.

4. In a paper published in the first volume of the *Transactions* Dr. Kasner established a connection between the theories of double binary and quaternary forms. In the present paper the author applies this connection to certain questions as to apolarity. A double binary form $f_{nm} = f(x_1, x_2; y_1, y_2)$ of the nth degree in $x_1 : x_2$ and the mth degree in $y_1 : y_2$, may be interpreted as the general algebraic curve of species (n, m) upon a quadric surface Q. When the two degrees are equal the curve is a complete intersection curve, and there corresponds to the form f_{nn} a unique surface of nth order F_n and also a unique surface of nth class ϕ_n. If two forms f_{nn} and $f'_{n'n'}$, where $n \geqq n'$, are apolar, then every surface of the n'th order through the curve $f'_{n'n'} = 0$ is apolar to the surface ϕ_n which corresponds to f_{nn}; conversely, if ϕ_n is apolar to any surface of n'th order through $f'_{n'n'} = 0$, then the double binary forms are apolar. This theorem, applied to the case where one of the forms is bilinear, reduces the problem of constructing all the curves on a quadric surface which are apolar to a conic to the corresponding problem of plane geometry. The discussion of apolar relations with respect to forms f_{nm} where $m \neq n$ occupies the final part of the paper. The method is based on the introduction of certain systems of polar forms and the related systems of surfaces.

5. The form of Green's theorem here used will be found in the Encyclopädie, II, *A7c*, pages 513, 514. Using the notation and terminology there explained, it is here proved that if throughout a certain region S of the (x, y)-plane the partial differential equation $L(u) = 0$ belongs to the elliptic type, and if u is a solution of this equation analytic throughout S, and if a function v exists continuous and having continuous first and second partial derivatives throughout S and satisfying the inequality $AM(v) \leqq 0$, then u cannot vanish at all points of a closed curve in S, unless it vanishes identically.

6. It is well known that if $f_0(y)$ and $f_1(y)$ are two functions which can be developed in power series about the point $y = 0$,

there exists one and only one solution $u(x, y)$ of Laplace's equation in two dimensions which, when $x = 0$, reduces to $f_0(y)$ while its partial derivative with regard to x reduces to $f_1(y)$, and that this function $u(x, y)$ can itself be developed in a power series about the point $x = 0, y = 0$. The chief theorem proved in the present paper is that if the series f_0 and f_1 both converge when $|y| < r$, the double power series for $u(x, y)$ converges within the square whose vertices are the points $(\pm r, 0)$, $(0, \pm r)$.

8. Miss Schottenfels's note on types of groups of order p^n, all of whose elements (I excluded) are of order p, contains a proof of the existence of a group of indices (modulo p) where p is a prime, and the generational definition of the abelian group G_{p^n} every element of which is of prime order p.

12. Dr. Ling's paper contains a treatment of a problem of a type suggested by Hermite, viz., that of the approximate representation of a function by means of algebraic functions. Padé, after showing the relation of the representation of functions by means of power series to the general problem, treated the case of functions defined by linear equations. In the present paper is considered the case of functions defined by quadratic equations of the type $y^2 + Qy + P = 0$, where Q and P are polynomials. After a definition of *reduced function* has been given, the paper treats of the determination of such functions in the general case and of the order of approximation to a specified function which is obtained in each case. The results obtained are generalizations of the two theorems proved by Padé.

13. Dr. Haskins's paper is an extension of the work in the author's paper on the invariants of quadratic differential forms to forms of higher degree. The methods used are the same, but the difficulties of the problem are much less. The result is as follows: The number of invariants of order μ for the general homogeneous differential form of degree m in n variables is

$$(n, \mu)\left\{(n, m) - \frac{n(n+\mu)}{\mu+1}\right\}, \quad (n, m) = \frac{(n+m-1)!}{(n-1)!\,m!}$$

The case of simultaneous invariants of several forms is also considered.

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SERIES WHOSE PRODUCT IS ABSOLUTELY CONVERGENT.

BY PROFESSOR FLORIAN CAJORI.

(Read before Section A of the American Association for the Advancement of Science, Pittsburg, July 1, 1902.)

§ 1. That two absolutely convergent series yield an absolutely convergent product was first shown by Cauchy.* About three quarters of a century later Alfred Pringsheim pointed out that an absolutely convergent product may result also from the multiplication of a conditionally convergent or even a divergent series by an absolutely convergent series.† That the product of two conditionally convergent series, or of a conditionally convergent series and a divergent series, or of two divergent series, may be absolutely convergent was first made public by the present writer.‡ Thereupon Alfred Pringsheim treated the subject from a more general point of view and, by very simple methods, showed that the property in question is typical of certain classes of series.§ The present writer developed a new class of series possessing this property, demonstrated the validity of the fundamental laws of algebra in the multiplication of infinite series, and generalized a theorem of Abel on the multiplication of series.‖ In the present article we aim to generalize some of the results previously obtained relating to absolutely convergent products of two or more series.

§2. In this investigation we shall start with an absolutely convergent series and determine pairs of series which are factors of the assumed series. Given the absolutely convergent series

$$U \equiv \sum_{n=0}^{n=\infty} u_n, \tag{1}$$

in which

$$u_n \equiv \sum_{r=0}^{r=n} c_r e_r c'_{n-r} e_{n-r},$$

* Analyse Algébrique, 1821, page 147.

† *Math. Annalen*, vol. 21 (1883), pp 357–359.

‡ *Transactions of the American Mathematical Society*, vol. 2, pp 25–36, January 1901; *Science*, new series, vol. 14, p. 395, September 13, 1901.

§ *Transactions of the American Mathematical Society*, vol. 2, pp. 404–412, October, 1901.

‖ BULLETIN, 2d series, vol. 8, pages 231–236, March, 1902.

e_r is the general term of any absolutely convergent series, say

$$e_r \equiv \frac{1}{(z+r)[\log(z+r)]^\lambda},$$

$\lambda > 1$, and c_r, c'_r are constants either real or complex. That series (1) is absolutely convergent becomes evident, if we observe that it is the product formed according to Cauchy's multiplication rule, of the two absolutely convergent series

$$\sum_{r=0}^{r=\infty} c_r e_r \quad \text{and} \quad \sum_{r=0}^{r=\infty} c'_r e_r.$$

Let us assume

$$a_{r-s} - a_r = c_r e_r, \tag{2}$$

$$h_r(b_{r-t} - b_r) = c'_r e_r,$$

where s and t are positive integers, a and b are real or complex numbers and h_r is an odd or an even power of -1. Let it be agreed that a and b cannot have negative subscripts; in other words, that $a_{-x} = b_{-x} = 0$. We have

$$u_n \equiv \sum_{r=0}^{r=n} c_r e_r \cdot c'_{n-r} e_{n-r} = \sum_{r=0}^{r=n} h_{n-r}(a_{r-s} - a_r)(b_{n-r-t} - b_{n-r}).$$

If we perform the indicated multiplications and collect the coefficients of a_r, we obtain

$$u_n \equiv \sum_{r=0}^{r=n} a_r(-h_{n-r}b_{n-r-t} + h_{n-r}b_{n-r} + h_{n-r-s}b_{n-r-s-t} - h_{n-r-s}b_{n-r-s}). \tag{3}$$

If we assume $h_n = -h_{n+1}$, we have

$$u_n \equiv \sum_{r=0}^{r=n} (a_{r-t} + a_r)(h_{n-r}b_{n-r} - h_{n-r-s}b_{n-r-s}). \tag{4}$$

It will be noticed that if, in the two terms in (3) which involve the factors a_r and a_{r-t}, respectively, we remove the parentheses, we obtain eight terms which are distributed among three terms

of the series in (4), namely the three terms which involve, respectively, the parentheses

$$(a_{r-2t} + a_{r-t}), \quad (a_{r-t} + a_r), \quad (a_r + a_{r+t}).$$

From the inspection of series (4) we readily see that the series (1) may be considered to be the product of the following two series:

$$\sum_{r=0}^{r=\infty} (a_{r-t} + a_r) \equiv a_0 + a_1 + \cdots + a_{t-1} + (a_0 + a_t) + (a_1 + a_{t+1}) + \cdots \tag{5}$$

and

$$\sum_{r=0}^{r=\infty} (b_r b_r - b_{r-s} b_{r-s}) \equiv b_0 b_0 + b_1 b_1 + \cdots + b_{s-1} b_{s-1} + (b_s b_s - b_0 b_0) + \cdots. \tag{6}$$

According to the condition $b_r = -b_{r+t}$, we are permitted to choose any sign we please for any t consecutive factors b_r. After such a choice has been made, the signs represented by any of the other factors b_r are determined.

Since relations (2) are the only conditions imposed upon the values of a and b, it is possible to choose these values so that each of the series (5) and (6) is absolutely convergent, conditionally convergent, or divergent. Thus, if $|a_r|$ is of the order of magnitude e_r, series (5) is absolutely convergent; if

$$|a_{r-t} + a_r| \geqq \frac{1}{(r+2)\log(r+2)},$$

but approaches the limit zero as r increases indefinitely and if $(a_{r-t} + a_r)$ is opposite in sign to and has greater numerical value than $(a_{r-t+1} + a_{r+1})$, then (5) is conditionally convergent; if $|a_{r-t} + a_r|$ does not approach the limit zero for all values of r, as r increases indefinitely, the series (5) is divergent. Similarly for (6). Yet in every case, the product of (5) and (6) is absolutely convergent.

Since, so far as we know, no two divergent series with *complex* terms have before been given, whose product is absolutely convergent, it may be well to construct a special example. Let

$a_r = b_r = A + B_r e_r$, where A and B_r are complex constants. Moreover, let $s = 1$ and $t = 2$, $h_0 = -h_1 = +1$. It will be seen that values which are not infinite can be assigned to the coefficients c_r and c'_r so that equations (2) are satisfied. By substitution in the two series (5) and (6) we obtain the following two series:

$$\left.\begin{aligned} S_1 \equiv{}& (A + B_0 e_0) + (A + B_1 e_1) + (2A + B_2 e_2 + B_0 e_0) \\ &+ (2A + B_3 e_3 + B_1 e_1) + \cdots, \\ S_2 \equiv{}& (A + B_0 e_0) - (2A + B_1 e_1 + B_0 e_0) - (B_2 e_2 - B_1 e_1) \\ &+ (2A + B_3 e_3 + B_2 e_2) + (B_4 e_4 - B_3 e_3) - \cdots. \end{aligned}\right\} \quad (7)$$

It will be seen that both series in (7) are divergent and complex, and that their product is absolutely convergent. Another pair of complex series possessing this property is given at the close of this article.

Since all the terms in the series S_1 are preceded by the positive sign, it is readily seen that any positive integral power of S_1 is a divergent series whose terms increase numerically without limit as r increases without limit. The same conclusion holds for the series S_2. Since $S_1 \cdots S_2 \cdots S_1 \cdots S_2 \cdots$ [to p pairs of factors] $= (S_1 S_2)(S_1 S_2) \cdots$ [to p parentheses] $= S_1^p \cdots S_2^p$, and since $S_1 \cdots S_2$ is an absolutely convergent complex product, it follows that the product of the two complex series S_1^p and S_2^p, the terms of both of which increase numerically without limit as r increases without limit, is absolutely convergent, no matter how large a value the integer p may have.*

If we let a_r and b_r be positive and decreasing monotonously toward zero in such a way that Σa_r and Σb_r are both divergent, if moreover, $t = 2$, $s = 1$, $h_0 = +1$, $h_1 = -1$, then (5) and (6) reduce to the two following series, one divergent, the other conditionally convergent, given by Pringsheim:†

$$\sum_{r=0}^{r=\infty} (a_{r-2} + a_r) \text{ and } \sum_{r=0}^{r=\infty} (-1)^{r+\left[\frac{r-1}{2}\right]} (b_{r-1} + (-1)^{r-1} b_r).$$

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* See BULLETIN, 2d series, vol. 8 (1902), pp. 233–236.

† See *Transactions of the American Mathematical Society*, vol. 2, p. 408, equation (B). The notation $\left[\frac{r-1}{2}\right]$ signifies here the largest integer contained in $\frac{r-1}{2}$.

In order to deduce from (5) and (6) the pair of conditionally convergent series, whose product is absolutely convergent, which we gave in the BULLETIN, volume 8 (1902), page 231, let $s = t = 4$; $h_0 = h_1 = h_2 = h_3 = +1$; $r = 4v, 4v+1, 4v+2$, or $4v+3$; $r' = 4v$. Let moreover the parenthesis $(a_{r-1} + a_r)$, which we represent for convenience by a'_r, be a real number which is positive when $r = 4v$ or $4v+1$ and negative when $r = 4v+2$ or $4v+3$, and such that $|a'_r| \leqq r^{-u}$, where $\frac{1}{2} < u \leqq 1$, and $\Sigma |a'_r|$ is divergent. Without violating conditions (2) we may assume further

$$a'_{r'} = a'_{r'+1}, \quad |a'_{r'+2}| - a'_{r'} = e_{r'} \pm w_{r'},$$
$$a'_{r'+2} = a'_{r'+3}, \quad |a'_{r'+2}| - a'_{r'+4} = e_{r'+4} \pm w_{r'+4},$$

where w_r is a quantity numerically not greater than

$$\frac{1}{(2+r)^{2-u}[\log(2+r)]^{\lambda}}.$$

Similarly we may assume, without violating conditions (2),

$$h_r b_r - h_{r-s} b_{r-s} \equiv b'_r, \quad |b'_r| \leqq r^{-u}, \quad \sum_{r=0}^{r=\infty} |b'_r| \text{ divergent},$$

b'_r a real number which is positive when $r = 4v$ and $4v+2$, and negative when $r = 4v+1$ and $4v+3$; also

$$b'_{r'} = b'_{r'+2}, \quad b'_{r'} - |b'_{r'+1}| = e_{r'} \pm w_{r'},$$
$$b'_{r'+1} = b'_{r'+3}, \quad b'_{r'+4} - |b'_{r'+1}| = e_{r'+4} \pm w_{r'+4}.$$

There result from these assumptions the two conditionally convergent series $\sum_{r=0}^{r=\infty} a'_r$ and $\sum_{r=0}^{r=\infty} b'_r$, whose product is an absolutely convergent series, which were previously given by us in the last mentioned article.

§7. If we are given a series

$$\sum_{r=0}^{r=\infty} (a_{r-1} + a_r),$$

such that $a_{r-s} - a_r = c_r c_{r'}$, it is quite evident that we can always find a mate for it, such that the product of the two shall be absolutely convergent.

Again it is possible to find two series, every term in the product of which, after the $(t+1)$th term, vanishes identically. Moreover, the two series may be so chosen that the product possesses the additional property of having for its sum any desired finite number N. To bring this about modify conditions (2) thus,

$$a_{r-s} - a_r = cd^r,$$

where d is any number, and, for $r \geqq t$,

$$h_r(b_{r-t} - b_r) = 0.$$

Then $\sum_{r=0}^{r=\infty} u_r$ is still absolutely convergent. Let t be even and

$$b_{t-1} = db_{t-2} = d^2 b_{t-3} = \cdots = d^{t-1} b_0,$$
$$h_0 = -h_1 = h_2 = -h_3 = \cdots = -h_t = 1.$$

Then we have, for $r \geqq t$,

$$u_r = b_0 cd^r - b_0 d \cdot cd^{r-1} + b_0 d^2 \cdot cd^{r-2} - \cdots - b_0 d^{t-1} \cdot cd^{r-t+1} \equiv 0.$$

Under the above conditions the two series (5) and (6) have a product, every term of which, after the $(t+1)$th term, vanishes identically.

To make the sum of the product of the two series (5) and (6) equal to N, place the sum of the first $t+1$ terms in the product equal to N and then determine the values of d which satisfy this condition.

Thus, let $t = 2$, $s = 1$, $c = +1$, then from $a_{r-1} - a_r = d^r$, for $r \geqq 1$, we get the series (5) (the first term of which is assumed to be a_0)

$$a_0 + (a_0 - d) + (2a_0 - d - d^2) + (2a_0 - 2d - d^2 - d^3) + (2a_0 - 2d - 2d^2 - d^3 - d^4) + \cdots.$$

Assuming $b_0 = 1$, the series (6) becomes

$$1 - (d+1) + (d-1) + (d+1) - (d-1) - (d+1) + (d-1) + \cdots.$$

All the terms in the product of these two series, after the second term, vanish. Putting the sum of the first two terms $= N = 0$, we have $a_0 - d(a_0 + 1) = 0$. If we assume $a_0 = -\frac{1}{9}^0$, then $d = 10$, and the two factor series become

$$U = -\frac{1}{9}^0 - 11\frac{1}{9} - 112\frac{2}{9} - \cdots$$
$$U' = 1 - 11 + 9 + 11 - 9 - 11 + 9 + 11 - \cdots$$

Since $U \cdot U' = 0$, we have $U^p \cdot U'^p = 0$. As all the terms in U are of the same sign, it is easily seen that U^p is divergent for all positive integral values of p. U' and U'^p are also divergent.

§ 8. If we assume $t = 2$, $s = 1$, $c = -1$, $a_0 = 1$, $b_0 = 1$, $N = 0$, then the condition that the sum of the product of (5) and (6) shall vanish becomes $d^2 + 1 = 0$ and (letting $i = \sqrt{-1}$) the factor series thus obtained are the two complex divergent series

$$1 + (1 + i) + i + 0 + 1 + (i + 1) + i + 0 + \cdots$$
$$1 - (1 + i) + (i - 1) + (i + 1) - (i - 1) - (i + 1) + \cdots$$

COLORADO COLLEGE, COLORADO SPRINGS,
April 12, 1902.

THREE SETS OF GENERATIONAL RELATIONS DEFINING THE ABSTRACT SIMPLE GROUP OF ORDER 504.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, October 25, 1902.)

1. CONSIDERABLE interest attaches to the simple group of order 504. The existence of this simple group was discovered by Professor Cole.* This was one of the facts that lead Professor Moore † to his investigation of the linear fractional group in the general Galois field, resulting in the discovery of a new doubly infinite system of simple groups.

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* "On a certain simple group," Mathematical Papers, Chicago Congress of 1893.

† BULLETIN, December, 1893; Mathematical Papers, Congress of 1893.

The first set of generational relations follows readily from a general theorem quoted, with references, in § 2. From it is derived, in §§ 5–9, the second set of generational relations

$$A^7 = I, \quad B^2 = I, \quad (AB)^3 = I, \quad (A^3BA^5BA^3B)^2 = I.$$

The same method was followed by the writer * to determine the well-known generational relations for the simple linear fractional groups in the $GF[5]$ and $GF[2^2]$, each of order 60; and for that in the $GF[7]$, of order 168. In a "Note on the simple group of order 504," Burnside † established the above second set by a direct analysis. Other derivations of this second set are due to R. Fricke ‡ and to De Séguier.§

The simplest of the three is the (new) third set

$$C^9 = I, \quad D^2 = I, \quad (CD)^3 = I, \quad (C^3DC^5D)^2 = I.$$

First Set of Generational Relations, §§ 2–4.

2. THEOREM.‖ *The group of all linear fractional substitutions of determinant unity in the $GF[p^n]$ is simply isomorphic with the abstract group generated by the operators T and S_λ (λ running through the series of p^n marks of the field), subject to the generational relations*

$$(1) \qquad T^2 = I, \quad S_0 = I, \quad S_\lambda S_\mu = S_{\lambda+\mu} \qquad (\lambda, \mu \text{ any marks}),$$

$$(2) \qquad S_\lambda T S_\mu T S_{\frac{\lambda-1}{\lambda\mu-1}} T S_{-(\lambda\mu-1)} T S_{\frac{\mu-1}{\lambda\mu-1}} T = I$$

(λ *and* μ *any marks such that* $\lambda\mu \neq 1$).

3. For $p^n = 2^3$, the group is a simple group of order 504. Let the $GF[2^3]$ be defined by the irreducible congruence

$$i^3 \equiv i + 1 \pmod 2,$$

so that the marks of the field are

$$0, \quad 1, \quad i, \quad i+1, \quad i^2, \quad i^2+1, \quad i^2+i, \quad i^2+i+1.$$

* Linear Groups, pp. 302–303.
† *Math. Annalen*, vol. 52 (1899), pp. 174–176.
‡ *Ibid.*, p. 335.
§ *Journal de Math.*, series 5, vol. 8 (1902), p. 267.
‖ Found, but not yet published, by Professor Moore. A different proof for finite fields is due to the writer: Linear Groups, pp. 300–302.

The relations $S_\lambda S_\mu = S_{\lambda+\mu}$ all follow from

$$(3)\quad S_1^2 = S_i^2 = S_{i+1}^2 = S_{i^2}^2 = S_{i^2+1}^2 = S_{i^2+i}^2 = S_{i^2+i+1}^2 = I,$$

$$(4)\quad S_1 S_i = S_{i+1},\ S_1 S_{i^2} = S_{i^2+1},\ S_1 S_{i^2+i} = S_{i^2+i+1},\ S_{i^2} S_i = S_{i^2+i}.$$

Taking the inverse of every member of (4) and applying (3), we get

$$S_i S_1 = S_{i+1},\ S_{i^2} S_1 = S_{i^2+1},\ S_{i^2+i} S_1 = S_{i^2+i+1},\ S_i S_{i^2} = S_{i^2+i}.$$

To show that $S_i S_\mu = S_{i+\mu}$ for every mark μ, we note that

$$S_i S_{i+1} = S_i \cdot S_i S_1 = S_1,\ S_i S_{i^2+1} = S_i \cdot S_{i^2} S_1 = S_{i^2+i} S_1 = S_{i^2+i+1}$$

$$S_i S_{i^2+i} = S_i \cdot S_i S_{i^2} = S_{i^2},\ S_i S_{i^2+i+1} = S_i \cdot S_{i^2+i} S_1 = S_{i^2} S_1 = S_{i^2+1}$$

Then $S_{i+1} S_\mu = S_{i+1+\mu}$ follows at once, since

$$S_{i+1} S_\mu = S_1 S_i S_\mu = S_1 S_{i+\mu} = S_{1+i+\mu}.$$

That $S_\lambda S_\mu = S_{\lambda+\mu}$ ($\lambda = i^2,\ i^2+1,\ i^2+i$, or i^2+i+1) follow quite similarly.

4. For $\lambda = 0$ or 1 or for $\mu = 0$ or 1, relation (2) reduces to *

$$(5)\quad (TS_1)^3 = I.$$

Among the remaining relations (2) occur the following six:

$$(6)\quad S_i T S_i T S_{i^2+i} T S_{i^2+1} T S_{i^2+i} T = I,$$

$$(7)\quad S_{i^2} T S_{i^2} T S_i T S_{i^2+i+1} T S_i T = I,$$

$$(8)\quad S_{i+1} T S_{i+1} T S_{i^2+1} T S_{i^2} T S_{i^2+1} T = I,$$

$$(9)\quad S_{i^2+1} T S_{i^2+1} T S_{i^2+i+1} T S_{i^2+i} T S_{i^2+i+1} T = I,$$

$$(10)\quad S_{i^2+i} T S_{i^2+i} T S_{i^2} T S_{i+1} T S_{i^2} T = I,$$

$$(11)\quad S_{i^2+i+1} T S_{i^2+i+1} T S_{i+1} T S_i T S_{i+1} T = I.$$

From these follow all the relations (2) in which neither λ nor μ is 0 or 1. For, in (6)–(11), each subscript i, $i+1$, i^2, i^2+1, i^2+i, i^2+i+1 is followed immediately by all of the others, excepting its reciprocal, provided the first subscript be regarded

* Thus, for $\lambda = 0$, (2) becomes $TS_\mu TS_1 TS_1 TS_{-\mu+1} T = I$. Transforming both members by TS_μ, we get $TS_1 TS_1 TS_1 = I$.

as following the last; while from (2) results an equivalent relation by using for the initial λ, μ any two consecutive subscripts. We have therefore the following

THEOREM.* *The abstract simple group of order* 504 *is generated by the operators* T, S_1, S_i, S_{i+1}, S_ϱ, $S_{\varrho+1}$, $S_{\varrho+i}$, $S_{\varrho+i+1}$ *subject to the generational relations* $T^2 = I$ *and* (3), (4), (5), (6), (7), (8), (9), (10), (11).

Second Set of Generational Relations, §§ 5–9.

5. Transforming both members of (8) by TS_{i+1}, we get

$$S_{i+1}TS_{i+1}TS_{i+1}TS_{i+1} = S_{\varrho+i}TS_\varrho TS_{\varrho+i}. \tag{12}$$

Denoting each product by P, we form PTP and get

$$S_{i+1}(TS_{i+1})^7 = S_{\varrho+i}TS_\varrho TS_{\varrho+i}TS_{\varrho+i}TS_\varrho TS_{\varrho+i}.$$

In view of (10), the second member equals

$$S_{\varrho+i}TS_\varrho T \cdot TS_\varrho TS_{i+1} \cdot S_{\varrho+i} = S_{i+1}.$$

Hence

$$(TS_{i+1})^7 = I. \tag{13}$$

Set $A \equiv TS_{i+1}$, $B \equiv S_i$. By (3), (4), (5), (13),

$$A^7 = I,\ B^2 = I,\ (AB)^3 = I. \tag{14}$$

6. From (10) and (7), we obtain respectively

$$S_\varrho TS_\varrho = S_{i+1}TS_{i+1}TS_iTS_{i+1}TS_{i+1} = A^{-2}BA^2,$$
$$S_{\varrho+i+1} = TS_iTS_\varrho TS_\varrho TS_iT.$$

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*Note added November 10. *The relations* (8), (9), (10) *may be omitted.* Indeed, we may write (6) thus:

$$S_iTS_iTS_{i^2+i+1}S_1TS_{i^2+1}TS_{i^2+i}T = I.$$

Replacing $TS_iTS_{i^2+i+1}$ by $S_{i^2}TS_{i^2}TS_iT$, as may be done by (7), and then TS_1T by S_1TS_1, as may be done by (5), we obtain (10), with its subscripts permuted cyclically. Again, we may write (6) thus:

$$S_iTS_{i+1}S_1TS_{i^2+i}TS_{i^2+1}TS_{i^2+i}T = I.$$

Replacing S_iTS_{i+1} by $TS_{i+1}TS_{i^2+i+1}TS_{i^2+i+1}T$, in virtue of (11), and then TS_1T by S_1TS_1, in virtue of (5), we obtain a relation immediately equivalent to (9). Finally, we may write (7) thus:

$$S_{i^2}TS_{i^2}TS_1 \cdot S_{i+1}TS_{i^2+i+1}TS_iT = I.$$

Replacing $S_{i+1}TS_{i^2+i+1}T$ by $TS_iTS_{i+1}TS_{i^2+i+1}$, in virtue of (11), and then TS_1T by S_1TS_1, we obtain (8).

But $ABA^{-1} = TS_1T$. Hence

$$S_{2i+i+1} = ABA^{-3}BA^3BA^{-1}, \tag{15}$$

$$S_{2i+1} = B \cdot ABA^{-3}BA^3BA^{-1} = A^{-1}BA^3BA^3BA^{-1}, \tag{16}$$

since $BAB = A^{-1}BA^{-1}$ by (14). Since S_{2i+1} is of period 2,

$$(A^3BA^3BA^3B)^2 = I. \tag{17}$$

7. We next express all the generators of the group in terms of A and B. Transforming (8) by TS_{2i+1}, we get

$$S_{2i+1}A^2TS_1S_{2i}TS_{2i}T = I.$$

Replacing S_{2i+1} and $S_{2i}TS_{2i}$ by their values and TS_1 by AB,

$$T = A^{-1}BA^3BA^3BA^2BA^3BA^2. \tag{18'}$$

Since $BA^2B = A^{-1}BA^{-2}BA^{-1}$ by (14), we get

$$\begin{aligned} T &= A^{-1}BA^3BA^2BA^{-2}BA^4BA^2 \\ &= A^{-1}BA^2BA^{-2}BA^{-3}BA^4BA^2. \end{aligned}$$

Replacing $BA^{-3}BA^4BA^2$, of period 2 by (17), by its inverse, we get

$$T = A^{-1}BA^2BA^3BA^3BA^3B. \tag{18}$$

8. The S_μ are given by (15), (16) and the relations

$$S_1 = TAB,\ S_{i+1} = T^{-1}A,\ S_{2i+i} = TABS_{2i+i+1},\ S_{2i} = S_{2i+i}B.$$

Some of these relations may be simplified. Thus,

$$\begin{aligned} S_{2i+i} &= A^{-1}BA^2BA^3BA^3BA^3B \cdot AB \cdot ABA^{-3}BA^3BA^{-1} \\ &= A^{-1}BA^2BA^3BA^3BA^{-1}BA^3BA^{-1} && \text{[by (14)]} \\ &= A^{-1}BA^2BA^3BA^4BA^4BA^{-1} && [BA^{-1}B = ABA] \\ &= A^{-2}BA^{-2}BA^3BA^4BA^4BA^{-1} && [BA^2B = A^{-1}BA^{-2}BA^{-1}] \\ &= A^{-2}BA^{-2}B \cdot BA^3BA^3BA^5 \cdot A^{-1} && \text{[by (17)]} \\ &= A^{-2}BABA^3BA^4 \\ &= A^{-3}BA^2BA^{-3} && [BAB = A^{-1}BA^{-1}]. \end{aligned}$$

Taking the inverse for S_{i+2}, we have the relations

$$(19)\quad \begin{cases} S_i = B,\ S_1 = TAB, & S_{i+1} = A^3BA^5BA^3, \\ S_{i+1} = T^{-1}A, & S_i = A^3BA^5BA^3B. \end{cases}$$

9. THEOREM. *The simple group of order* 504 *is generated by two operators* A *and* B *subject to the generational relations* (14) *and* (17).

It remains to show that the operators defined by (15), (16), (18) and (19) satisfy the relations $T^2 = I$, (3)–(11), in view of relations (14) and (17) alone. We derive at once the relations

$$S_i^2 = I,\ S_{i+i+1}^2 = I,\ S_{i+1}^2 = I,\ S_i^2 = I,\ S_iS_i = S_{i+2}$$

$$S_{i+i}^2 = A^3BA^5 \cdot BA^{-1}B \cdot A^5BA^3$$

$$= A^3BA^{-1}BA^{-1}BA^3 = A^7 = I.$$

Since $BA^2B = A^{-1}BA^{-2}BA^{-1}$ by (14), we get

$$T^2 = (A^5BA^5BA^3BA^3BA^3B)^2$$

$$= A^5BA^5BA^2 \cdot A^2BA^4BA^4 \cdot A^5BA^2BA^3BA^3B \qquad [\text{by (17)}]$$

$$= A^5BA^5 \cdot BA^4BA^4BA^2B \cdot A^2BA^3BA^3B$$

$$= A^5BA^5 \cdot A^5BA^3BA^3 \cdot A^2BA^3BA^3B \qquad [\text{by (17)}]$$

$$= (A^5BA^3BA^3B)^2 = I \qquad [\text{by (17)}].$$

That $S_{i+1} \equiv T^{-1}A$ has period 2 is shown as follows:

$$ATAT = BA^3BA^3BA^3BA^5BA^3BA^3BA^3B \qquad [\text{by (18)}]$$

$$= BA^2 \cdot A^3BA^4BA^4 \cdot A^3BA^3BA^3B = I \qquad [\text{by (17)}].$$

We may show that $S_1^2 = I$ by indicating the square of

$$(20)\qquad T^{-1}AB = BA^4BA^4BA^4BA^5BA^2B$$

and replacing $BA^{-1}B$ by ABA six times in succession. We have therefore derived all the relations (3) and $T^2 = I$. Next,

$$S_1S_i = TAB \cdot B = T^{-1}A = S_{i+1};$$

$$S_1 = TAB = A^{-1}BA^3BA^3BA^2BA^5BA^3B = S_{i+1}S_i,$$

upon employing for T formula (18′) which follows from (18), (17), (14). Hence $S_1S_{i^2} = S_{i^2+1}$. Again,

$$S_1S_{i^2+i} = S_1S_{i^2}S_i = S_{i^2+1}S_i = (S_iS_{i^2+1})^{-1} = S_{i^2+i+1}^{-1} = S_{i^2+i+1}.$$

Hence relations (4) have all been derived.

It remains to derive relations (5)–(11). From what precedes, we may make use of relations (1). Since $TS_1 = AB$, (5) follows from (14). We next verify that

$$S_{i^2}TS_{i^2} = A^{-2}BA^2. \tag{21}$$

Employing $S_{i^2} = S_{i^2}^{-1} = BA^4BA^2BA^4$ and the value of T from (18), we obtain as the condition for (21),

$$A^2BA^4BA^2BA^3BA^2BA^3BA^5BA^2B = I.$$

Replacing the first two products BA^2B by $A^{-1}BA^{-2}\ BA^{-1}$, we get

$$A^2BA^3BA^{-2}BABA^{-2}BA^2BA^5BA^2B = I.$$

Replacing BAB by $A^{-1}BA^{-1}$ and then $BA^4BA^4BA^2$ by its inverse, in view of (17), we get

$$A^2BABA^3BABA^2B = I,$$

a relation following readily from (14). As in §6, (10) and (7) follow from (21), (15), (16), and

$$ABA^{-1} = AS_{i+1}S_iS_{i+1}A^{-1} = TS_iT. \tag{22}$$

From (18′) and (21) follows (8) as in §7. Note that (10) also follows from (8), and $A^7 = I$, as in §5.

Employing (22), relation (6) may be written

$$B\cdot ABA^{-1}\cdot S_{i^2+i}\cdot TS_1\cdot S_{i^2}TS_{i^2}\cdot S_iT = I.$$

Applying (14), (19) and (21), the relation becomes

$$A^{-1}BA^{-2}\cdot A^3BA^5BA^3\cdot AB\cdot A^{-2}BA^2\cdot BT = I.$$

By (14) we may replace BA^5B and $BA^{-2}B$ by ABA^2BA:

$$A^{-1}BA\cdot ABA^2BA\cdot A^4\cdot ABA^2BA\cdot A^2BT = I.$$

Replacing BA^6B by ABA, we get, by (18),

$$A^{-1}BA^2BA^3BA^3BA^3BT = T^2 = I.$$

Transforming (11) by T, we get

$$TS_{i+1} \cdot S_{i+2} TS_{i+2} \cdot S_{i+1} T \cdot S_{i+1} T \cdot S_i \cdot TS_{i+1} = I.$$

Setting $TS_{i+1} = A$ and $S_i = B$, and applying (21), it becomes an identity.

To verify (9), we write it in the form

$$S_{i+1} \cdot TS_1 \cdot S_{i+2} TS_{i+2} \cdot S_{i+1} T \cdot S_i \cdot S_{i+2} TS_{i+2} \cdot S_{i+1} T = I.$$

Applying (16), (21), $TS_1 = AB$, $S_{i+1} T = A^{-1}$, $S_i = B$, it becomes

$$A^{-1} BA^3 BABABA^{-2} BA = I,$$

which is an identity since $BABAB = A^{-1}$ by (14).

Third Set of Generational Relations, §§10–12.

10. Consider the two operators

$$C = S_i T, \quad D = S_{i+1}.$$

From (6) and (8) we obtain respectively

$$(S_i T)^4 = S_{i+2} TS_{i+1} TS_{i+2} T, \quad TS_{i+1} TS_{i+2} T = S_{i+1} TS_{i+1} TS_{i+1}.$$

Hence, applying also (11), we derive

$$(S_i T)^9 = (S_{i+i+1} TS_{i+1} TS_{i+1})^3 S_i T = I.$$

Also, $DC \equiv S_1 T$ is of period 3 by (5). Hence

$$D^2 = I, \; C^9 = I, \; (DC)^3 = I. \tag{23}$$

11. Noting that $C^{-1} DC = TS_{i+1} T$, we may write (11) thus:

$$S_{i+i+1} TS_{i+i} \cdot DC \cdot D \cdot C^{-1} \cdot C^{-1} DC = I,$$

$$S_{i+i+1} TS_{i+i} = C^{-1} DC^2 DC^{-1} D = C^{-1} DC^3 DC.$$

Multiplying the latter on the left by S_{i+1} and on the right by $S_i T$, we get

$$S_{i+2} TS_{i+2} T = DC^{-1} DC^3 DC^2 = CDC^4 DC^2.$$

Applying this result in (7), we get

$$S_{i+i+1} T = C^6 DC^5 DC^7.$$

We now obtain the formulæ *

* It may be shown that $S_{i+i+1} = DC^7 DC^3 DC^7$.

$$S_{\varrho+i} = S_{\varrho+i+1} T \cdot TS_1 = C^6DC^5DC^6D, \tag{24}$$

$$S_{\varrho+1} = S_{\varrho+i} S_{i+1} = C^6DC^5DC^6, \tag{25}$$

$$S_\varrho T = S_{i^2+i}^{-1} S_i T = DC^3DC^4DC^4. \tag{26}$$

Since $S_{\varrho+1}$ is of period 2, it follows from (25) that

$$(DC^5DC^3)^2 = I. \tag{27}$$

We may write (6) in the form

$$C^2 \cdot S_2 T \cdot TS_i \cdot TS_\varrho \cdot S_1 T \cdot S_{\varrho+i} \cdot T = I.$$

Applying (26) twice and (24) once, this relation gives

$$C^2DC^3DC^4DC^{-1}DC^5DC^4DC^5DC^6D = T.$$

Replacing $DC^{-1}D$ by CDC in view of (23) and then C^3DC^5D by DC^4DC^6, in view of (27), we get

$$T = C^6DC^3DC^4DC^5DC^6D. \tag{28}$$

From the definitions of A and B at the end of § 5, we get

$$A = TD = C^6DC^3DC^4DC^5DC^6, \tag{29}$$

$$B = CT^{-1} = CDC^3DC^4DC^5DC^6DC^3. \tag{30}$$

12. THEOREM. *The simple group of order* 504 *is generated by two operators* C *and* D, *subject to the generational relations* (23) *and* (28).

In view of the theorem of § 9, it suffices to show that, if A and B are defined by (29) and (30), relations (14) and (17) follow from relations (23) and (27). First,

$$BA = CT^{-1} \cdot TD = CD, \tag{31}$$

so that BA is of period 3 in view of (23). To show that $B^2 = I$, we replace DC^6DC^4 by its inverse C^5DC^3D in the indicated square of B and get

$$B^2 = CDC^3DC^4 \cdot DCD \cdot C^6DC^4DC^5DC^6DC^3.$$

Replacing DCD by $C^{-1}DC^{-1}$ and, in the result, DC^3DC^5 by C^4DC^6D, we get

$$B^2 = CDC^7DCDC^5DC^6DC^3.$$

Making similar replacements, we get $B^2 = C^6DC^9DC^3 = I$.

In the indicated product AA, we replace DC^5DC^3 by C^6DC^4D, then DCD by $C^{-1}DC^{-1}$, then C^6DC^4D by DC^5DC^3, then $DC^{-1}D$ by CDC twice, and get

$$A^2 = C^7DC^8. \tag{32}$$

It follows at once that

$$A^6 = C^7DC^6DC^6DC^8.$$

It will equal the expression for A^{-1} given by (29) if

$$C^4DC^6DC^6DC^5 = DC^4DC^5DC^6D.$$

Replacing C^4DC^6D by DC^3DC^5, the condition becomes

$$I = C^4DC^7DCDC^5DC^6D.$$

Replacing DCD by $C^{-1}DC$ and then C^6DC^4D by DC^5DC^3, the condition reduces to an identity in view of $D^2 = I$, $C^9 = I$.

Finally, to derive (17), we note that

$$BA^3BA^5BA^3 = BA \cdot A^2 \cdot BA \cdot A^4 \cdot BA \cdot A^2,$$

and substitute the values of BA and A^2 given by (31) and (32). The result $CDC^5DC^4DC^8$ is evidently of period 2 by (23).

Generators of the Linear Fractional Group of Order 504.

13. For the first set of generational relations, we may take

$$T: \quad z' = \frac{1}{z}; \qquad S_\mu: \quad z' = z + \mu.$$

Hence, for the second set, we may take

$$A: \quad z' = \frac{1}{z} + i + 1; \qquad B: \quad z' = z + i.$$

For this second set, Burnside employs

$$A': \quad z' = iz; \qquad B': \quad z' = \frac{z + i^2}{i^2 z + 1}.$$

We readily obtain the relations

$$A' = TS_{i+1}TS_iTS_iTS_iTS_{i+1} = S_{i^2+i+1}TS_{i^2+i}TS_{i+1}TS_{i+1},$$

$$B' = S_{i+1}TS_{i+1}TS_{i^2+i+1}TS_{i^2+i}.$$

For the third set, we may evidently take

$$C: \quad z' = 1 \div (z + i); \qquad D: \quad z' = z + i + 1.$$

Note that $S_{i+1}T$ is of period 7; while

$$BA^3\colon\quad z'=\frac{z+i+1}{z}$$

is of period 9; also, as a check on (27), that

$$DC^3DC^3\colon\quad z'=\frac{z+i^2}{iz+1}.$$

THE UNIVERSITY OF CHICAGO,
October 19, 1902.

GENERATIONAL RELATIONS DEFINING THE ABSTRACT SIMPLE GROUP OF ORDER 660.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, October 25, 1902.)

THE abstract group of order 660, simply isomorphic to the group of linear fractional substitutions of determinant unity taken modulo 11, may be generated by two operators S and T subject to the following relations:*

(1) $$S^{11}=I,\quad T^2=I,\quad (ST)^3=I,$$

(3 3 3 3 3), (7 7 7 7 7), (2 2 4 8 4), (4 4 9 7 9),
(5 5 2 9 2), (6 6 8 9 8), (8 8 5 3 5), (9 9 10 8 10),
(2 3 9 6 7), (2 7 6 9 3), (2 8 3 7 10), (2 10 7 3 8),
(3 6 4 5 10), (3 10 5 4 6), (4 7 5 6 10), (4 10 6 5 7),

where the symbol $(a\ b\ c\ d\ e)$ denotes the relation

$$S^aTS^bTS^cTS^dTS^eT=I.$$

These relations are very redundant. We proceed to reduce

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*For a proof of the general theorem due to Moore, see Dickson, Linear Groups, § 278, Corollary. When the field is of order a prime p, we may set $S_1=S$, $S_i=S^i$. By § 279, we need retain only the 16 relations (a, b, c, d, e) given above, since the others follow by a cyclic permutation of a, b, c, d, e. It may be shown that S^4T, S^2T, S^4T, S^5T, S^6T, S^7T, S^8T, S^9T have the respective periods 12, 5, 5, 6, 6, 5, 5, 12, results not used here.

the nineteen to five simple relations. For use in this reduction we note that from (1) follows

$$(2) \qquad TST = S^{-1}TS^{-1}, \quad TS^{-1}T = STS,$$

$$(3) \qquad TS^2T = S^{-1}TS^{-2}TS^{-1}, \quad TS^{-2}T = STS^2TS.$$

Applying to (2 2 4 8 4) relation (3_1), we get

$$STS^{-2}TS^3TS^8TS^4T = I.$$

$$S^3TS^8T = TS^2TS^{-1}TS^7 = TS^3TS^8,$$

in view of (2_2). Hence (2 2 4 8 4) may be replaced by

$$(4) \qquad (S^3TS^8T)^2 = I.$$

By (2 2 4 8 4) we have

$$S^7TS^9T = TS^8TS^4TS^2.$$

Substituting this value in (4 4 9 7 9), we get

$$(5) \qquad (S^4TS^6T)^2 = I.$$

To (5 5 2 9 2) we apply (3_1) twice in succession. Transforming the result by S, we obtain (4 4 9 7 9).

In (6 6 8 9 8) we replace S^6TS^6T by $TS^2TS^9TS^2$ given by (5 5 2 9 2). The resulting relation becomes an identity upon applying (2_2) twice.

Transforming (8 8 5 3 5) by S^7 and twice replacing TS^5TS^7 by its inverse S^4TS^6T, in view of (5), we obtain $(ST)^5 = I$.

In (9 9 10 8 10) we apply (2_2) three times in succession and obtain an identity.

Transforming (2 3 9 6 7) by S^5 and replacing S^8TS^3T and TS^7TS^5 by their inverses, given by (4) and (5), we get

$$TS^8TSTSTS^4T = TS^7(ST)^3S^4T = I.$$

Transforming (2 7 6 9 3) by S^3 and replacing S^5TS^7T and TS^3TS^8 by their inverses, we get

$$TS^4TSTSTS^8T = TS^3(ST)^3S^8T = I.$$

To (2 8 3 7 10) we apply (2_2) and transform the result by S^{-1}. There results relation (4).

Applying (2_2) to (2 10 7 3 8), we get (4).

In (3 6 4 5 10) we replace TS^6TS^4 by its inverse and get

$$(S^{-1}T)^3 = I.$$

Similarly, (3 10 5 4 6), (4 7 5 6 10), and (4 10 6 5 7) follow by applying first (5) and then (1).

Transforming (3 3 3 3 3) by S^6 and replacing S^8TS^3T by its inverse TS^6TS^3, given by (4), we get

$$TS^8TS^6TS^3TS^3TS^6 = I,$$

a relation following by taking the inverse of (8 8 5 3 5).

Transforming (7 7 7 7 7) by S^2 and replacing S^5TS^7T by its inverse TS^4TS^6, given by (5), we get

$$TS^4TS^2TS^7TS^7TS^2 = I,$$

a relation following by taking the inverse of (4 4 9 7 9).

Hence the simple group of order 660 *is generated by two operators S and T subject to relations* (1), (4), *and* (5).

THE UNIVERSITY OF CHICAGO,
October 22, 1902.

THE CARLSBAD MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG, SEPTEMBER, 1902.

THE annual meeting of the Deutsche Mathematiker-Vereinigung was held at Carlsbad, September 21–26, forming as usual a part of the Versammlung Deutscher Naturforscher und Aertze. At the business meeting of the Society, committees were appointed to make arrangements for the international mathematical congress which will be held at Heidelberg in August, 1904. A committee was also appointed to consider the feasibility of forming at some point in Germany a very complete collection of mathematical literature. Professor Gutzmer resigned from the position of secretary of the society, and Professor Krazer was elected his successor. The meeting of the Society will be held next year at Cassel.

The following papers were read at the meeting:

Professor CZUBER, Vienna: "A theorem in the theory of errors, and its application."

Professor GRÜNWALD, Prague: "Derivatives with variable index as analytic functions of the latter, and a related functional operation."

Professor HASKELL, California: "The representation of certain resultants as determinants."

Dr. JAHNKE, Berlin: "An elementary theory of the theta functions of one and two arguments."

Dr. JAHNKE, Berlin: "The relation of Steiner to Jacobi."

Professors KLEIN, Göttingen, and MEYER, Königsberg: "The progress on the Encyklopädie der mathematischen Wissenschaften."

Professor KOHN, Vienna: "On a certain sextuple of rays."

Professor KOWALEWSKI, Greifswald: "A report on Lie's theory of transformation groups."

Professor KOWALEWSKI, Greifswald: "On a criterion of du Bois-Reymond for the development in Fourier series."

Professor KOWALEWSKI, Greifswald: "A new characteristic of the projective group of a norm curve."

Dr. LIEBMANN, Leipsic: "On the conformal representation of convex surfaces upon the sphere."

Professor MEYER, Königsberg: "On the spheres inscribed in a tetrahedron."

Professor E. MÜLLER, Vienna: "On the theory of linear systems of curves and surfaces of the second order."

Professor F. MÜLLER, Steglitz: "The abbreviations of the titles of mathematical publications."

Professor SCHUBERT, Hamburg: "Enumerative relations for incidence and coincidence in linear space of higher dimensions."

Dr. STEINITZ, Berlin: "A report on polyhedra."

Dr. VON STERNECK, Vienna: "On an analogy to the additive number theory."

Professor WÄLSCH, Brünn: "On binary analysis."

Professor Czuber developed the relations which hold for a linear form of n independent errors of observation, when the errors follow singly the exponential law, and applied the results to direct observations of equal exactness.

Professor Grünwald spoke on his researches in the theory of the derivatives with variable index, and their development in powers of the latter. The coefficients in the development are found by linear functional operations, and are represented by definite integrals. The application to the solution of certain linear differential equations was mentioned.

Sylvester's dialytic method, applied to finding the resultant of forms of more than two variables, gives a determinant of too high an order — the resultant with a foreign factor. Cayley has found the resultant in the form of a determinant without foreign factor for the case of three ternary quadratic forms, and Hesse for the case of three ternary forms of which two are quadratic and one linear, and of four quaternary forms of which three are quadratic and one linear. These results follow (those of Hesse less directly) from the theorem, that if n homogeneous forms in n variables vanish simultaneously, then the Jacobian vanishes, and if the forms are of the same degree, the first derivatives of the Jacobian vanish also. Professor Haskell adds to this theorem that, if the forms are of the same degree, the second derivatives of the Jacobian are proportional to certain linear functions of the second derivatives of the forms, and the application of this theorem gives the resultant of four or of five quadratic forms, homogeneous in four or five variables respectively, in the form of a determinant without foreign factor.

Dr. Jahnke spoke on Caspary's theory of the theta functions and gave some further developments. In Jacobi's lectures the theory of the theta functions was developed from their definition by series of exponential functions. Caspary used this definition but gave the theory a purely algebraic character without function-theoretical considerations, introducing the orthogonal system with Euler's form for the coefficients in order to express the many algebraic relations between the thetas. For further development of the theory only the algebraic and differential identities between the elements of an orthogonal system are required. The introduction to the elliptic transcendentals, as given by Caspary, appears elementary enough for the use of students of physics and technology.

In his second paper Dr. Jahnke read some recently discovered letters of Steiner and Jacobi.

Professors Klein and Meyer spoke on the progress of the Encyklopädie. The last part of Volume I containing the index will probably appear before Easter. The chief difficulty in Volume II has been overcome, but further publication will not be immediate. Volume III, on geometry, is divided into three

main parts, and articles from each part will appear in the next few months; in Part I, Enriques, Principles of Geometry; in Part II, Dingeldey, Conic Sections; in Part III (Differential Geometry), Mangold, Space Curves and Surfaces. Volume IV (Mechanics) is progressing satisfactorily. Publication will soon begin on Volume V (Physics), with Bryan's article on Thermodynamics, and also on Volume VI (Geodesy and Geophysics).

Professor Kohn's paper is in abstract as follows: There is a single system of six lines, such that every interchange is produced by a collineation or correlation. These lines may be called an equianharmonic ray sextuple from the fact that the four points of intersection of every line that meets four of them are equianharmonic. If an icosahedron form be interpreted on a cubic space curve the six chords given by the six "pairs" of the icosahedron form are such a system; the six corresponding axes form a second system which is related to the first in many ways. Interesting results follow from the fact that two surfaces of second order, one containing three of the lines of the system, the other the remaining three, cut each other in the edges of a skew four-edge. Corresponding to the ten cases where the six lines reduce to two sets of three, are ten tetrahedra, each of which is inscribed in and circumscribed about every other. The forty vertices and forty sides form a Witting configuration, and it follows that the Klein group of 61 collineations is a subgroup of the hyperelliptic collineation group under which the Witting configuration is invariant. The above is in close relation to results of Klein, Witting, Maschke, and Burkhardt.

Professor Kowalewski's first paper was a report on Lie's theory of transformation groups. Much of Lie's work has a value apart from his group theory, *e. g.*, the idea of element and element union, important in the theory of differential equations; that of contact transformations, which Lie conceived in its generality in the first years of his activity; and, of greatest importance, the infinitesimal transformation. Lie was in possession of most of these ideas in 1873 when he began to develop his theory of transformation groups, above all, however, that of the infinitesimal transformation. He had found in 1872 that an ordinary differential equation of first order with a known

infinitesimal transformation could be integrated by quadrature, and had proposed to himself the problem: Of what value for the integration of a system of differential equations is the knowledge of infinitesimal transformations which leave the system unaltered? He considered the case of a complete system of linear partial differential equations in detail. If two infinitesimal transformations Xf and Yf leave such a system unchanged, then this holds also for the "Klammerausdruck"

$$X(Yf) - Y(Xf) = (XY).$$

By this theorem the question is reduced to the consideration of a complete system, with r known infinitesimal transformations $X_1f, \cdots, X_rf$ which satisfy relations of the form

$$(X_iX_k) = \sum C_{ik_s}X_sf,$$

and these relations hold for all point transformations. This was the starting point for the theory of transformation groups. Lie found by involved reckoning that r independent infinitesimal transformations which satisfy relations of the above form, belong to a continuous group with r parameters, and showed conversely that every such group contains r independent infinitesimal transformations which satisfy the above relations. He had then, although without rigorous proofs, the two fundamental theorems of his group theory. The first presentation of Lie's results is in an article by Klein and Lie in the *Göttinger Nachrichten*, December 3, 1874. The differential equations of his later first fundamental theorem occur here, while the third, which relates to the constants C_{ik_s}, appeared in 1876 in the Norwegian *Archiv*. In the Göttingen note Lie indicates some applications of his theory to ordinary differential equations, which he did not complete until 1882 (*Archiv for Mathematik*, volumes 8 and 9). In 1882 Lie applied his theory of finite groups to infinite groups by considering only such groups as can be defined by differential equations. He showed also that every such group determines infinitely many differential invariants (*Mathematische Annalen*, volume 24).

A norm curve in R_n, *i. e.*, a rational curve of order n which is contained in no plane point manifold of R_n, admits a group of ∞^3 projective transformations. Professor Kowalewski, in his second paper, gave a complete enumeration of all projective transformation groups which contain the above as a subgroup.

For $n > 2$ there exists in general (except for the group of all projective transformations) only one such group. The single exception is in case of R_6, where a fourteen group exists, which has the Killing constitution and plays a part in the work of Engel.

In his third paper, Professor Kowalewski calls attention to a little known criterion for the convergence of a Fourier series which was published by P. du Bois-Reymond in 1881 in the *Comptes Rendus* and earlier in *Crelle*. A simpler proof of this criterion was indicated.

Closely related to the theorem that a closed convex surface can not in general be deformed (verbogen), without introducing singularities, is the following theorem: A closed convex surface can not in general be conformally represented, without singularities, upon the sphere. Dr. Liebmann considered a special surface of revolution which admits at most a three parameter group of regular conformal transformations, while for the sphere the six parameter group of homographic transformations forms such a regular conformal group. The surface cannot, therefore, be represented conformally upon the sphere without singularities, and the theorem holds for this special surface. Other such examples can be found showing that the theorem in general holds.

A theorem of Schwarz states that if on each face of a tetrahedron circumscribing a sphere lines are drawn from the point of tangency to the vertices, then the corresponding angles about the points of tangency on the four faces are equal. Interpreting this theorem by projective geometry, Professor Meyer gave some interesting generalizations.

Projective geometry is generalized after Klein by considering the properties of geometrical configurations invariant under a more general transformation group. Of special importance is the projective geometry of curves and surfaces of the second order. A simple theorem of this geometry may contain many results from the standpoint of the ordinary projective geometry. Professor E. Müller showed this by applying the theorem, that in a web of conics (Kegelschnittgewebe) two families always have a common conic, to two special webs,

first, one with a double point, and second, one whose curves are tangent to two lines. Three conics ϕ_1, ϕ_2, ϕ_3, of which ϕ_3 has double tangency with ϕ_1, determine a web of the first kind, and from the theorem follows that in the family (ϕ_3, ϕ_2) there are three conics each of which passes through a pair of opposite vertices of the tangential complete quadrilateral common to ϕ_1 and ϕ_2. The theorem of Darboux follows when ϕ_3 is a point pair, and theorems of Chasles on confocal conics result through metric specialization. The application of the above general theorem to the second web also gives interesting theorems, *e. g.*, the point pairs of two projective point rows of the plane belong to a web of this kind. The fruitfulness of similar considerations for higher dimensions was indicated.

Professor F. Müller spoke on the abbreviations of the titles of mathematical journals, and presented a list of some 600 proposed abbreviations. This list will be manifolded for the present, and eventually published.

In the enumerative geometry only those incidence formulæ have been generalized to linear space of n dimensions, by Schubert, Pieri and others, which relate to a line and incident point. Professor Schubert now attacks the general problem, and finds all the fundamental incidence and coincidence formulæ for two linear spaces of dimensions n and $n + m$.

Dr. Steinitz gave a report on the general theory of the polyhedron, starting from the results of Euler. The symbolic table of Catalan, the work of Eberhardt on polyhedron "fields" and the general question of polyhedra of multiple connectivity were treated at some length.

In the additive number theory the number of ways is sought in which a given number can be formed by the addition of given elements. An analogous question is the following: If n and M are given numbers, in how many ways can a number congruent to n (mod. M) be formed additively from the elements $1, 2, \cdots, M-1$. Stern (1863) studied the question for prime moduli. In the paper of Dr. von Sterneck the general problem was considered, with following results: If $(n)_i^0$ is the number of such representations of n (mod. M) by i elements, and $(n)_i$ the similar number when zero may appear as an element, then

$$(n)_i^0 = (n)_i - (n)_{i-1} + (n)_{i-2} - \cdots + (-1)^i (n)_0,$$

$$i(n)_i = \sum (n-e)_{i-1} - \sum (n-2e)_{i-2} + \cdots + (-1)^i \sum (n-ie)_0.$$

In the summations e assumes all values of a complete remainder system with respect to M, and $(n)_0 = 0$ except when $n \equiv 0 \pmod{M}$, in which case $(n)_0 = 1$.

The difference between the numbers of even and odd representations of n is then

$$\Delta(n) = \sum (-1)^i (n)_i^0.$$

In particular, for $M \equiv p^k$,

$$\Delta(n) = 0, \quad -p^{k-i}, \quad p^k - p^{k-i}$$

according as n is not divisible by p^{k-i}, is divisible by p^{k-i}, or is divisible by p^k.

For any $M = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ the theorem holds: If n contains any p_i to a power $\leqq a_i - 2$, then $\Delta(n) = 0$. If n contains the factors $p_1, p_2, \cdots, p_i$ to the powers $a_1 - 1, a_2 - 1, \cdots, a_i - 1$, and the factors $p_{i+1}, \cdots, p_k$ to the powers $a_{i+1}, \cdots, a_k$, then

$$\Delta(n) = (-1)^i \frac{\phi(M)}{(p_1 - 1)(p_2 - 1) \cdots (p_i - 1)}.$$

If n contains all the factors of M to the same degree, then $\Delta(n) = \phi(M)$. $\phi(M)$ is the Euler function.

Further relations can be developed from the theorem: The difference $\psi(M)$ between the number of odd and even representations when the H. C. F. of the elements is relative prime to the modulus, satisfies the equation $\sum \psi(d) = \phi(M)$, where d takes on the values of all factors of M, and is then the number theory derivative of the Euler function $\phi(M)$.

Professor Wälsch spoke on the application of the theory of binary forms to geometry and mechanics. By the introduction of the Kugelkreis, points, lines, and planes are represented by binary forms. A point a lies in a plane b when the second "Ueberschiebung" of the form a with b is unity. A vector can also be represented by a binary form, and the first and second "Ueberschiebungen" of two forms give the two vector products. A rational space curve of order n is given by the form $r_y^n s_x^2 / c_y^n$. In applying the binary form to mechanics, the

expressions for the ellipse of inertia, kinetic energy, and the equations of motion, take very simple forms.

I wish to thank the speakers for the use of manuscripts, and to express my indebtedness to many who gave me abstracts of their papers.

C. M. MASON.

GÖTTINGEN,
October, 1902.

SHORTER NOTICES.

Compte rendu du deuxième Congrès international des Mathématiciens tenu à Paris du 6 au 12 août 1900. Procès-verbaux et Communications publiés par E. DUPORCQ. Paris, Gauthier-Villars, 1902. 8vo. 455 pp.

THE members of the second international congress of mathematicians may have felt a little annoyance at the delay in the publication of the official report; but now that this is at last in their hands, the delay no longer appears unreasonable. A glance through the volume makes it plain that the editor's task has been very serious, inasmuch as its execution has depended on the good pleasure of so many contributors, and these so widely dispersed, that it has been quite beyond his power to expedite matters. Editor and publisher are to be congratulated on the beautiful volume which has recently appeared; it was worth waiting for.

The first 26 pages contain the minutes of the sectional and general meetings; pages 27–153 are occupied by the four principal addresses, with which is placed also a French translation of Hilbert's survey of the future problems of mathematics; pages 154–450 are devoted to the sectional papers. Of these thirty-two are given, thus leaving six or eight unreported for reasons beyond the editor's control. In general the papers appear in a French version, the exceptions being one in Italian and eight in English. As a matter of fact, the majority of the communications were made in French, but there were certainly some in German.

The most interesting articles for present reading are naturally those of a somewhat general character, for the results of more special interest have not been held back pending this publica-

tion. The principal addresses have already been given in outline in the BULLETIN,* with the exception of Mittag-Leffler's account of the correspondence between Weierstrass and S. Kowalevski. This is perhaps as interesting as anything in the book. At the end of the volume there is given, in a French translation, a paper by Veronese, "Les postulats de la géométrie dans l'enseignement," which was intended for the Congress, though it was not actually presented then. Among the shorter articles the most satisfying and appropriate are those of the character of an aperçu, as for instance d'Ocagne, "Sur les divers modes d'application de la méthode graphique à l'art du calcul"— Padé, "Aperçu sur les développements récents de la théorie des fractions continues"— and Amodeo, "Coup d'œil sur les courbes algébriques au point de vue de la gonalité." Amodeo, in particular, has given a most interesting résumé of results obtained regarding groups of points, g_k^1, cut out on a curve of order m by adjoints of order $m-3-a$, adjoints whose existence presupposes the condition $p \geqq \frac{1}{2}am+1$. But different readers will naturally find most to interest them in different papers.

CHARLOTTE ANGAS SCOTT.

Probabilités et Moyennes Géométriques. Par EMANUEL CZUBER. Traduit par HERMANN SCHUERMANS. Préface de CHARLES LAGRANGE. Paris, A. Hermann, 1902. xi + 244 pp.

THE various problems considered in the direct theory of probability may be roughly divided, as regards the number of possible cases, into four classes. First, the ordinary problems arising in connection with games of chance, where the number of cases is limited, the methods employed being those of combinatorial analysis. Second, problems where the number of cases is still finite, but very large, so that recourse must be made to approximate results based upon Stirling's formula. Here belong, for example, Bernouilli's theorem and the so-called law of large numbers. The third class includes those problems in which the number of cases is infinite, depending upon the values of a certain number of arbitrary parameters; while in a fourth class we may place the problems depending upon arbitrary laws or functions. The last class has as yet received little attention. It may be remarked in passing that

* BULLETIN, vol. 7 (1900-1901), pp. 57-79.

there is another type, intermediate between the second and third, in which the number of cases is infinite, but forms merely a denumerable assemblage. Such problems arise naturally in connection with the ordinary theory of numbers, but so far as the reviewer knows have never been considered.

Professor Czuber's well known monograph, now appearing in an attractive French translation, is devoted to the third class of problems, and the related questions of mean value. The term geometric or local probability is justified, as is remarked in the introduction, in view of the fact that if a problem depends upon arbitrary parameters, it is possible to translate it into an equivalent geometric problem relating to points or more complicated configurations taken at random in certain regions. The three chapters on local probability treat respectively 1° Points taken at random on lines or surfaces or in space; 2° Lines taken at random in a plane or in space; 3° Planes taken at random. The discussion of mean value, forming the second part of the book, is much briefer, occupying only a single chapter of sixty pages.

The questions considered in this youngest branch of the theory of probability have for their type the famous needle problem of Buffon, proposed and solved by the latter in his Essai d'arithmétique morale, published 1777 in the fourth volume of the Supplément a l'histoire naturelle, but composed some years earlier. Laplace in his Théorie, pages 359–362, treated the same problem, without however referring to Buffon. It may be remarked that the latter proposed also a more complicated problem, where the plane is ruled into congruent rectangles by means of two sets of parallels, but his solution is incorrect. Laplace gave the solution for the case where the needle or rod is shorter than both dimensions of the rectangle. Todhunter in his History of the mathematical theory of probability, § 650, considers the more general problem, but his results, according to Czuber, are incomplete. Since Laplace the chief contributions to the subject are due to Barbier, Jordan and Lalanne of the French school, and McColl, Sylvester and Crofton of the English school.

Of especial interest are the accounts, in connection with some of the problems discussed by Professor Czuber, of experiments verifying the theoretical results. Thus in connection with Buffon's problem, Professor Wolf in 1850, using a rod four fifths the distance between the parallels, found that in 5000 trials

there were 2532 cases of intersection. The theoretical probability in this case is $8/5\,\pi$, so that the probable value of π, obtained by comparison with the experimental probability, is 3.1596. No mention however is made of a similar series of experiments reported by De Morgan in his Budget of Paradoxes, page 172, leading to the still more accurate value 3.1412. It would be of interest if some one would undertake to carry out the experimental rectification of curves suggested on page 116 of Czuber's book, and based upon Crofton's theorem that the number of random lines which intersect a closed convex curve is proportional to the length of the curve.

It is a commonplace to remark that in no branch of mathematics is it so easy to make errors as in the theory of probability. The development of the theory is marked by a continual revision of the results of earlier investigators. Especially is this true of geometrical probability, where the essential difficulty lies in the definition of what is to be regarded as equally possible, or, as Venn happily expresses it, in "the idealization of the randomness." Mathematically this amounts to fixing the independent parameters which are to be regarded as equicrescent in the problem considered. In the present book only one error of any importance has come to the notice of the reviewer. This occurs in finding the probability that three segments, each less than a given limit, shall determine an acute or obtuse triangle respectively (pages 29, 30). The results obtained, $1 - \pi/4 = .216$ and $\pi/4 = .784$, cannot both be correct since there is a third case, not considered in the text, where the three segments do not determine any triangle. The error is not corrected in the translation, so it may be worth while to give here the correct results, which are $1 - \pi/4 = .216$, $\pi/4 - \frac{1}{2} = .285$, $\frac{1}{2} = .500$, respectively, for the three cases mentioned.

The English student finds introductions to the subject of local probability in the excellent chapters in Todhunter's and Williamson's Integral Calculuses (the latter chapter written by Professor Crofton); but of course these cannot compare in completeness with a monograph. The present translation should therefore prove welcome to the large class of students who find it more economical to read French than German.

E. KASNER.

Annuaire des Mathématiciens, 1901–1902 ; publié sous la direction de MM. C.-A. LAISANT, AD. BUHL. Paris, C. Naud, 1901. 8vo. xxiv + 469 pp.

THE idea of publishing an annual list of those interested in mathematics was brought forward with much earnestness by Professor F. Rudio at the first international congress of mathematicians at Zurich in 1897. The matter was not taken up, however, until two years later, when M. Laisant, recognizing the favorable opportunity afforded by the international congress to be held at Paris, presented the proposition to M. Naud, the publisher. The plan contemplated an alphabetic list of living mathematicians, with their leading titles, the mathematical societies to which they belong, and their addresses. For the purposes of the compilation a mathematician was defined as a person belonging to one of the following groups: (1) Members of mathematical societies, or of mathematical sections of general scientific societies; (2) authors of mathematical works of an original nature; (3) those who teach mathematics as their special department of work. As a result of the labors of the compilers, more than six thousand names appear in the register.

The work also contains lists of mathematical societies, of associations having mathematical sections, and of the journals devoted to this science. A brief necrology of mathematicians, covering the years 1900 and 1901, is made the more valuable by an added biography of Hermite. Unfortunately all dates are omitted in the necrology, a fault which will probably be remedied in future editions. The work closes with a half dozen "notices scientifiques," mostly foreign to the subject in hand:—By Paul Appell: "Sur le principe de la moindre contrainte de Gauss":—By J. Petersen: "Les 36 officiers"—By A. G. Greenhill: "Les fonctions elliptiques au point de vue de leurs applications."—By Ch. Méray: "La langue internationale auxiliaire *Espéranto* et la littérature scientifique."—By P. H. Schoute: "Le nombre des points, des droites, des plans, etc., contenus dans un hyperespace linéaire."—By P. H. Schoute: "La *Revue semestrielle des publications mathématiques*."

The general plan of the work is very commendable. Such a publication puts scientific workers more closely in touch with one another, and makes the mathematical world seem more real. It places at the student's hand a mass of information

that it has heretofore been exceedingly difficult to reach, information especially welcome to those having editorial duties to perform.

Unfortunately, however, the list has been so hastily compiled as to admit a number of errors that are, to say the least, unfortunate. Were it not that this is the initial number, that new ground is being broken, and that the task of condensing several thousand replies was so great, reviewers would be justified in seriously criticising the whole work. As it is, they will doubtless rest content with pointing out certain lines for improvement, hoping, as they are probably justified in doing, that the second number will remedy the defects of the first.

To justify the intimation of inaccuracy a few types of the errors contained in the work may be mentioned. From the list it appears that Lord Kelwin lives in Glascow, England; that there is a Brogham Young College in Utah, and a Haward University at Cambridge; that Professor Byerly's middle name is Elvood, and that the well-known Münster mathematician is Professor Kiliing. Several prominent names are omitted entirely, including W. S. Burnside, W. W. R. Ball, A. W. Panton, and E. M. Langley, while, as if to atone for this neglect, the name of Professor P. F. Smith appears twice on the same page, and that of the late Professor Craig appears both in the list of living mathematicians and in the necrology. The names of the American states are frequently omitted, and occasionally the state is given without the city. In the list of imperfections should also be mentioned the omission of the initials of a large number of French mathematicians, a national habit but an international annoyance.

It does not seem too much to hope that the second volume will be critically revised by a good linguist, that an effort will be made to enlarge the list judiciously, that the necrology will contain the dates of birth and death, that the list of periodicals will be made more nearly complete, that the notes will be submitted for correction to the various persons concerned, and that the extraneous matter will give place to topics more germane to the subject in hand.

DAVID EUGENE SMITH.

NOTES.

THE approaching Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY will extend through the two days, Monday and Tuesday, December 29–30. The Council will meet on Monday morning, and the annual election of officers and other members of the Council will be held on Tuesday morning. At the opening of the afternoon session on Monday, the retiring President, Professor ELIAKIM HASTINGS MOORE, will deliver the presidential address, the subject of which will be: "The Foundations of Mathematics."

ON recent academic occasions, the AMERICAN MATHEMATICAL SOCIETY has been represented by delegates as follows: At the Abel Centenary, by Professor E. B. VAN VLECK and Dr. E. B. WILSON; at the installation of President James, of Northwestern University, by President E. H. MOORE; at the installation of President Swain, of Swarthmore College, by Professor CHARLOTTE ANGAS SCOTT. The Society has recently received an invitation to send delegates to the coming centenary of the University of Jurievi (Dorpat).

THE forthcoming report of the librarian of the AMERICAN MATHEMATICAL SOCIETY will show that the library has made satisfactory progress during the year. There are now about twice as many periodicals on the exchange list as there were a year ago, about three times as many bound volumes of a non-periodical character, and over twenty times as many bound volumes of journals. The total number of bound volumes has increased from 121 to over 900. Several noteworthy gifts have been made to the library, including Bowditch's translation of Laplace's Mécanique céleste, and two volumes by the late Professor Benjamin Peirce from his own collection. An appeal is again made to members to present to the library copies of their works, whether elementary or advanced, to the end that the collection may become an exponent of the country's mathematical progress. It is also suggested that members may have copies of earlier works, including volumes of journals, which they would be willing to donate for such a purpose. Such gifts would be very welcome, especially if published in this country. Since the Society has no funds with which to make

purchases, it is forced to depend solely upon the generosity of its friends and upon its accessions in the way of exchanges.

THE National academy of sciences held its autumn meeting at Johns Hopkins University on November 11 and 12. No mathematical papers were presented.

AT the meeting of the London mathematical society held on November 13, the following papers were read:—By Dr. E. W. HOBSON: "On the infinite and the infinitesimal in mathematical analysis."—By Professor D. HILBERT: "On the proposition concerning the equality of the base angles in an isosceles triangle."—By Professor A. C. DIXON: "Expansion by means of Lamé's functions."—By Mr. W. H. YOUNG: "Note on unclosed sets of points defined as the limits of a sequence of closed sets of points."—By Professor H. LAMB: "Wave propagation in two dimensions."—By Professor M. J. M. HILL: "The continuation of certain fundamental power series."—By Mr. L. CRAWFORD: "A geodesic on a spheroid and an associated ellipse."

AN appeal has been sent out by the treasurer of the Royal society for financial support in completing the author catalogue of scientific papers from 1883 to 1900, and in publishing the subject index for the nineteenth century. The list of papers, arranged by authors, has been completed to 1883, at an expense of nearly $75,000, and the work on the list arranged by subjects has been partly done. The work of cataloguing papers issued since the beginning of the twentieth century has become so burdensome that the Royal Society has turned it over to an international committee.

THE second international congress of mathematicians will be held at Heidelberg in 1904.

COLLEGE OF FRANCE. — The following mathematical courses are given during the first semester of the present academic year, opening December 1, 1902:—By Professor CAMILLE JORDAN: Differential equations, two hours.—By Professor M. BRILLOUIN: Electric waves, two hours.—By Professor J. HADAMARD: Calculus of variations, one hour; Differential equations of continuous media, one hour.—By Professor LEBESGUES: Definite integrals and Fourier series, one hour.

It is of interest to recall that the chair now held by Professor Lebesgues was established a few years ago by M. Peccot under

the peculiar condition that the appointee should be under thirty years of age. The tenure of office is three years. M. Lebesgue's predecessor was M. E. Borel, whose lectures on Divergent series (1899–1900), Series with positive terms (1900–1901), and Meromorphic functions (1901–1902), recently published by Gauthier-Villars, present substantial evidence of the wisdom of M. Peccot's foundation. Both M. Borel and M. Lebesgue are graduates of the Ecole normale supérieure.

AT the Ecole normale supérieure, Professor J. TANNERY lectures on Galois theory and Professor P. Painlevé on Abelian functions.

PROFESSOR J. W. GIBBS, of Yale University, has been elected a corresponding member of the Munich academy of sciences.

MR. R. TUCKER retired from the position of honorary secretary of the London mathematical society on November 13. He has held this office since 1867, having been elected soon after the organization of the society.

A PORTRAIT of the late Professor P. G. Tait was unveiled at Peterhouse, Cambridge University, in October, by Lord Kelvin. Addresses were made by Lord Kelvin and by Sir George Stokes, master of Pembroke.

MR. R. W. H. T. HUDSON, B.A., has been appointed lecturer in mathematics at University College, Liverpool.

DR. H. S. CARSLAW lecturer in mathematics in the University of Glasgow, and fellow of Emmanuel College, Cambridge, has been appointed professor of pure and applied mathematics at the University of Sydney.

PROFESSOR MARTIN DISTELI, of Carlsruhe, has been called to the University of Strassburg as associate professor of mathematics.

DR. K. DÖHLEMANN, of Munich, has been advanced to the associate professorship in descriptive geometry.

PROFESSOR E. MÜLLER, of Königsberg, Prussia, has been called to the Technische Hochschule, at Vienna, as professor of mathematics.

DR. M. RADAKOVIC, of Innsbruck, has been advanced to an associate professorship of mathematics.

PROFESSOR D. KIKUCHI, president of the University of Tokio, recently created a baron by the emperor, has been made minister of education for Japan. Dr. SADAJI TAKAGI, of Tokio, has been advanced to an associate professorship of mathematics.

PROFESSOR F. KOLÁČEK, of Brunn, has been called to Prague as professor of mathematical physics in the Bohemian University.

PROFESSOR W. H. BOUGHTON, of Denison University, has been appointed professor of civil engineering in the University of West Virginia.

THE death is announced of Professor NIKOLAUS BUDAJEW, of St. Petersburg, at the age of sixty-nine years.

GENERAL ANNIBALE FERRERO, director of the Italian geodetic survey, died on August 7, in Rome.

PROFESSOR P. H. PHILBRICK died at Medford, Oregon, October 10, 1902. Dr. J. W. DAVIS died in New York, November 7, 1902. Both were members of the AMERICAN MATHEMATICAL SOCIETY.

THE following catalogues of second-hand mathematical works have recently appeared: R. Friedlaender & Sohn, 11 Carlstrasse, Berlin, N. W., no. 420, general mathematics, 114 pages; no. 421, calculus of probabilities, 16 pages. Mayer and Müller, 2 Prinz Louis Ferdinandstrasse, Berlin, N. W., no. 189, history of mathematics and other sciences, 5295 numbers; A. Herrman, 6 Rue de la Sorbonne, Paris, no. 76, part 2, mathematical works from the library of the late Joseph Bertrand, 88 pages.

PROFESSOR G. W. GREEN, of the department of mathematics in the Illinois Wesleyan University, died at Bloomington, Ill., on December 10, aged forty-five years.

PROFESSOR HENRY MITCHELL, formerly of the Massachusetts Institute of Technology and well known for his work on the U. S. Coast and Geodetic Survey, died in New York on December 1, at the age of seventy-two.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ABEL (N. H.). Mémorial publié à l'occasion du centenaire de sa naissance. Kristiania [1902]; Leipzig, Teubner. 4to. 13 + 119 + 135 + 61 + 64 + 59 + 2 pp, 2 plates, 6 facs. M. 21.00

BENOIST (A.). See CLEBSCH (A.).

BLICHFELDT (H. F.). On the determination of the distance between two points in space of n dimensions. 4to. 1902. (*Transactions of the American Mathematical Society* 3, pp. 467–481.)

BLUM (R.). Cykloiden und Cykloidalen als Umhüllungskurven und deren Zusammenhang mit den Fusspunktkurven der Kegelschnitte. (Progr.) Stuttgart, 1902. 4to. 56 pp.

BOHLMANN (G.). See SERRET (J. A.).

CLEBSCH (A.). Leçons sur la géométrie, recueillies et complétées par F. Lindemann et traduites par A. Benoist. (En 3 volumes.) Vol. I: Traité des sections coniques et introduction à la théorie des formes algébriques. Nouveau tirage. Paris, Gauthier-Villars, 1903. 8vo. Fr. 12.00

CZUBER (E.). Wahrscheinlichkeitsrechnung und ihre Anwendung auf Fehlerausgleichung, Statistik und Lebensversicherung. 1ste Hälfte. Leipzig, Teubner, 1902. 8vo. Pp. 1–304. (Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Vol. IX, 1.) M. 12.00

DUNKEL (O.). Regular singular points of a system of homogeneous linear differential equations of the first order. 8vo. 1902. (*Proceedings of the American Academy of Arts and Sciences* 38, pp. 341–370.)

EMCH (A.). Algebraic transformations of a complex variable realized by linkages. 4to. 1902. (*Transactions of the American Mathematical Society* 3, pp. 493–498.)

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band III, 3. Heft 1: H. von Mangoldt, Anwendung der Differential- und Integralrechnung auf Kurven und Flächen; R. von Lilienthal, Die auf einer Fläche gezogenen Kurven. Leipzig, Teubner, 1902. 8vo. Pp. 1–183.

ENGEL (F.). See GRASSMANN (H.).

ESTANAVE (E.). Nomenclature des thèses des sciences mathématiques, soutenues en France dans le courant du 19e siècle devant les Facultés des sciences de Paris et des départments. Paris, Gauthier-Villars, 1902. 8vo. Fr. 2.00

Revue décennale des thèses présentées à la Faculté des sciences de Paris, en vue du grade de docteur ès sciences, du 1er janvier 1891 au 31 décembre 1900, avec l'indication des périodiques contenant la plupart des ces mémoires ou leurs analyses. Paris, Gauthier-Villars, 1901. 8vo. 115 pp. Fr. 5.00

FRICKE (R.). Hauptsätze der Differential- und Integralrechnung, als Leitfaden zum Gebrauch bei Vorlesungen zusammengestellt. 3te, umgearbeitete Auflage. Braunschweig, Vieweg, 1902. 8vo. 15 + 218 pp. M. 5.00

GOTTSCHALK (A.). Die konforme Abbildung gewisser krummlinig begrenzter Vielecke. II. (Progr.) Münster, 1902. 8vo. 20 pp. M. 1.20

GRASSMANN (H.). Gesammelte mathematische und physikalische Werke. Auf Veranlassung der mathematisch-physischen Klasse der k. sächsischen Gesellschaft der Wissenschaften herausgegeben von F. Engel. Vol. II. Teil 2: Die Abhandlungen zur Mechanik und zur mathematischen Physik, herausgegeben von J. Lüroth und F. Engel. Leipzig, Teubner, 1902. 8vo. 8 + 266 pp. M. 14.00

GROSSMAN (M.). Ueber die metrischen Eigenschaften kollinearer Gebilde. Frauenfeld, 1902. 4to. 27 pp. M. 1.50

HAMBURGER (M.). Gedächtnisrede auf Immanuel Lazarus Fuchs, geboren am 5. V. 1833, gestorben am 26. IV. 1902. Mit dem Bildnis des Verstorbenen, sowie einem Verzeichnis seiner Schriften. Leipzig, Teubner, 1902. 8vo. 16 pp. M. 1.00

HARNACK (A.). See SERRET (J. A.).

HAUSSNER (R.). See SCHERING (E.).

HOEFER (F.). Histoire des mathématiques, depuis leurs origines jusqu'au commencement du 19e siècle. 5e édition. Paris, Hachette, 1902. 16mo. 3 + 609 pp. Fr. 4.00

HUTCHINSON (J. I.). See SNYDER (V.).

JAHRBUCH über die Fortschritte der Mathematik, begründet von C. Ohrtmann. Herausgegeben von E. Lampe und G. Wallenberg. Vol. 31: 1900. Heft 2. Berlin, Reimer, 1902. 8vo. Pp. 481–672. M. 6.00

KEYSER (C. J.). Mathematical productivity in the United States. 8vo. 1902. (*Educational Review* 1902, pp. 346–357.)

KOWALEWSKI (G.). See PFAFF (J. F.).

LAMPE (E.). See JAHRBUCH.

LANDFRIEDT (E.). Theorie der algebraischen Funktionen und ihrer Integrale. Leipzig, Göschen, 1902. 12mo. 4 + 294 pp. Cloth. (Sammlung Schubert, No. XXXI.) M. 8.50

——. Thetafunktionen und hyperelliptische Funktionen. Leipzig, Göschen, 1902. 12mo. 4 + 155 pp. Cloth. (Sammlung Schubert, No. XLVI.) M. 4.50

LIEBMANN (H.). See LOBATSCHEFSKIJ (N. I.).

LILIENTHAL (R. VON). See ENCYKLOPÄDIE.

LINDEMANN (F.). See CLEBSCH (A.).

LOBATSCHEFSKIJ (N. I.). Pangeometrie. (Kasan, 1856.) Uebersetzt und herausgegeben von H. Liebmann. Leipzig, Engelmann, 1902. 12mo. 95 pp. Cloth. (Ostwald's Klassiker der exakten Wissenschaften, No. 130.) M. 1.70

LÜROTH (J.). See GRASSMANN (H.).

MANGOLDT (H. VON). See ENCYKLOPÄDIE.

NOETHER (H.). See RIEMANN (B.).

PETERSEN (J.). Analytisk plangeometri. 4te udgave. Del I. Kjobenhavn, 1902. 8vo. Pp. 1–96. M. 2.50

PFAFF (J. F.). Allgemeine Methode, partielle Differentialgleichungen zu integriren. (1815.) Aus dem Lateinischen übersetzt und herausgegeben von G. Kowalewski. Leipzig, Engelmann, 1902. 12mo. 84 pp. Cloth. (Ostwald's Klassiker der exakten Wissenschaften, No. 129.) M. 1.40

PICKLER (A.). Ueber die Auflösung der Gleichung $\phi(x) = n$, wenn $\phi(m)$ die Anzahl derjenigen Zahlen bezeichnet, welche relativ prim zu m und kleiner als m sind. (Progr.) Wien, 1901. 8vo. 17 pp.

REISENHOFER (R.). Die sphärischen Kegelschnitte. (Progr.) Kremsier, 1902. 8vo. 8vo. 9 pp.

RIEMANN (B.). Gesammelte mathematische Werke. Nachträge, herausgegeben von M. Nöther und W. Wirtinger. Leipzig, Teubner, 1902. 8vo. 8 + 116 pp. M. 6.00

SAUERBECK (P.). Einleitung in die analytische Geometrie der höheren algebraischen Kurven nach den Methoden von J. P. de Gua de Malves. Ein Beitrag zur Kurvendiskussion. Leipzig, Teubner, 1902. 8vo. 6 + 166 pp. (Abhandlungen zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, begründet von M. Cantor, Heft 15.) M. 8.00

——. Der Satz von de Gua über die Wendepunkte der Kurven dritter Ordnung. (Progr.) Reutlingen, 1901. 8vo. 8 pp.

SCHERING (E.). Gesammelte mathematische Werke. Herausgegeben von R. Haussner und K. Schering. (In 2 Bänden.) Vol. I. Berlin, Mayer & Müller, 1902. 4to. 8 + 412 pp. M. 25.00

SERRET (J. A.). Lehrbuch der Differential- und Integralrechnung. Mit Genehmigung des Verfassers deutsch bearbeitet von A. Harnack. 2te, durchgesehene Auflage, herausgegeben von G. Bohlmann. Vol. III. Lieferung 1: Differentialgleichungen, herausgegeben von G. Bohlmann und E. Zermelo. Leipzig, Teubner, 1903. 8vo. Pp. 1–304. M. 6.00

SNYDER (V.) and HUTCHINSON (J. I.). Differential and integral calculus. New York, American Book Co. [1902]. 12mo. 16 + 320 pp. Cloth. (Modern mathematical series.) $2.50

STEININGER (T.). Studien zu Hesse's Analytischer Geometrie der geraden Linie, des Punktes und des Kreises in der Ebene. (Progr.) Rosenheim, 1902. 8vo. 38 pp., 1 plate.

STOLIAROV (N.). Collection of exercises in higher mathematics. Part I: Exercises in the differentiation of functions. Kiev, 1902. 8vo. 116 pp. (Russian.) M. 4.00

VALLÉE-POUSSIN (C. J. DE). Cours d'analyse infinitésimale. (En 2 volumes.) Vol. I. Paris, Gauthier-Villars, 1903. 8vo. 14 + 372 pp. Fr. 12.00

WALLENBERG (G.). See JAHRBUCH.

WERTHEIM (G.). Anfangsgründe der Zahlenlehre. Mit den Bildnissen von Fermat, Lagrange, Euler und Gauss. Braunschweig, Vieweg, 1902. 8vo. 12 + 427 pp. M. 9.00

WIRTINGER (W.). See RIEMANN (B.).

ZERMELO (E.). See SERRET (J. A.).

II. ELEMENTARY MATHEMATICS.

AGAFOV (D. V.). Arithmetical formulæ and their application to the solution of problems. 2 parts. Orenburg, 1902. 8vo. 64 + 79 pp. (Russian.) M. 4.00

ALASIA (C.). I complementi di geometria elementare. Milano, Hoepli, 1902. 16mo. 15 + 244 pp. (Manuali Hoepli.)

BAUSCHINGER (C. W.). See HAUCK (A. F.).

BERKENBUSCH (H.). Mathematisches Uebungsbuch für Realschulen, Realprogymnasien und Progymnasien. Aufgaben aus den Abgangsprüfungen sechsstufiger höherer Lehranstalten, zusammengestellt und mit Resultaten versehen. Berlin, Simion, 1902. 8vo. 5 + 136 + 41 pp. M. 2.80

BORK (H.). Mathematische Hauptsätze. Ausgabe für Gymnasien. Nach dem Tode des Verfassers herausgegeben von M. Nath. Teil I: Pensum der Unterstufe (bis zur Untersekunda einschliesslich). 4te, durchgesehene und den preussischen Lehrplänen von 1901 angepasste Auflage. Leipzig, Dürr, 1903. 8vo. 200 pp. Cloth. M. 2.10

——. Ausgabe für Realgymnasien und Oberrealschulen. Nach dem Tode des Verfassers herausgegeben von M. Nath. Teil I: Pensum der Unterstufe (bis zur Untersekunda einschliesslich). Nach der 3ten, vom Verfasser besorgten Auflage durchgesehene und den preussischen Lehrplänen von 1901 angepasste Ausgabe. Leipzig, Dürr, 1903. 8vo. 242 pp. Cloth. M. 2.50

BURILLO DE SANTIAGO (M.). Programa de primer curso de matemáticas que comprende nociones de arithmética y geometría. Madrid, 1902. 8vo. 21 pp. (Istituto general y técnico de San Isidro de Madrid.)

CERUELO Y OBISPO (J.). See MORENO REY (S.).

COOPER (G. H.). Elementary arithmetic of the octimal notation. San Francisco, Cal., Whitaker & Ray Co., 1902. 12mo. 70 pp. (Western mathematical series, No. 2.) $0.25

CRAIG (E. S.). See MARCOU (C. A.).

F. (F.). Arithmétique (cours moyen). Paris, Poussielgue [1902]. 16mo. 305 pp.

FARISANO. Elementi di geometria descrittiva secondo i programmi dei reali istituti tecnici. Roma, Forzani, 1902. 8vo. 132 pp.

FISCHER (F.). See HAUCK (A. F.).

FOURREY (E.). Récréations arithmétiques. Paris, Nony, 1902. 8vo. Fr. 3.50

FUSS (K.). Resultate und Andeutungen zur Auflösung der Aufgaben aus der Buchstabenrechnung und Algebra. Für Schulen und zum Selbstunterricht bearbeitet. 5te, vermehrte und verbesserte Auflage. Nürnberg, Korn, 1902. 8vo. 7 + 176 pp. M. 1.60

——. Sammlung von Konstruktions- und Rechenaufgaben aus der Planimetrie und Stereometrie. Mit vielen, vollständig gelösten Beispielen. Für den Schul- und Selbstunterricht bearbeitet. 3te, vermehrte und verbesserte Auflage. Nürnberg, Korn, 1902. 8vo. 8 + 252 pp. M. 2.50

GEORGES (N. DE). See REBIÈRE (A.).

GIDEON (E.). The model algebra; arranged for elementary schools. Philadelphia, Eldredge, 1902. 12mo. 149 pp. Cloth. $0.60

GLAUER (R.). Die trigonometrische Aufgabe in Untersekunda. (Progr.) Erfurt, 1902. 8vo. 20 pp., 13 plates.

HAUCK (A. F.) und HAUCK (H.). Lehrbuch der Arithmetik. Mit zahlreichen Beispielen und Uebungsaufgaben. (In 3 Teilen.) Teil I, Abteilung 2: Für Real-, Gewerb- und Handelsschulen. 8te, durchgesehene und verbesserte Auflage, herausgegeben von C. W. Bauschinger. 4 + 191 pp. Teil II, Abteilung 1: Für Latein-, Real- und Handelsschulen. 6te, umgearbeitete Auflage, herausgegeben von F. Fischer. 6 + 228 pp. Nürnberg, Korn, 1903. Cloth. M. 5.30

HEREDIA (G. DE). Lecciones de trigonometría, ajustada estrictamente al programa de ingreso en la Escuela central de ingenieros industriales. Madrid, Velasco, 1902. 8vo. 95 pp. Fr. 5.00

HOČEVAR (F.). Lehrbuch der Arithmetik und Algebra nebst einer Sammlung von Uebungsaufgaben für Oberrealschulen. Leipzig, Freytag, 1902. 8vo. 2 + 274 pp. Cloth. M. 3.60

HOLZMANN (A.) und MASSINGER (R.). Geometrische Anschauungslehre, im Anschluss an den Lehrplan der badischen Realschulanstalten. (In 3 Teilen.) Teil I. Erster Abschnitt der ebenen Gebilde. Pensum der Klasse V. 3te Auflage. Karlsruhe, Reiff, 1902. 8vo. 32 pp. M. 0.60

——. Teil II: Zweiter Abschnitt der ebenen Gebilde. Pensum der Klasse IV. 3te Auflage. Karlsruhe, Reiff, 1902. 8vo. 30 pp. M. 0.60

HOLZMÜLLER (G.). Elemente der Stereometrie. Teil 4: Fortsetzung der schwierigeren Untersuchungen; Berechnung und stereometrische Darstellung von statischen Trägheits- und Centrifugalmomenten homogener Raumgebilde; Simpsonsche Regel; verallgemeinerte Schichtenformel; gewisse Zuordnungen und konforme Abbildungen im Dienste solcher Bestimmungen; Nachtrag über das Katenoid, seine Krümmungsverhältnisse und sphärische Abbildung und über seinen Zusammenhang mit der Gausschen Pseudosphäre und der Minimalschraubenregelfläche. Leipzig, Göschen, 1902. 8vo. 11 + 311 pp. M. 9.00

JIMENEZ RUEDA (C.). Lecciones de geometría métrica. (In 2 volumes.) Madrid, Suárez, 1902. 8vo. 215 + 248 + 18 pp., 34 plates. Fr. 16.00

JOCELYN (L. P.). An algebra for high schools and academies. Philadelphia, Butler, Sheldon & Co. [1902]. 12mo. 445 pp. Cloth. $1.25

KRYZHANOVSKY (V. A.). Solutions of 256 geometric exercises by A. Davidov. 3d edition, revised. Kiev, 1901. 8vo. 137 pp. (Russian.) M. 3.00

MACÉ DE LÉPINAY (A.). See VACQUANT (C.).

MARCOU (C. A.) and CRAIG (E. S.). Oxford questions in algebra; papers set in pass moderations; with answers. 2d series, from Michaelmas term 1891 to Trinity term 1899. London, Simpkin, 1902. 12mo. Boards. 2s. 6d.

MARKS (C. I.). Mathematical questions and solutions from the *Educational Times*. Vol. II. New series. London, Hodgson, 1902. 8vo. Cloth. 6s. 6d.

MARTÍNEZ GARCÍA (M.). Programa de geometría. Madrid, 1902. 8vo. 20 pp. (Instituto del Cardenal Cisneros.) Fr. 1.00

——. Programa de nociones y ejercicios de arithmética y geometría. Primer curso. Madrid, 1902. 8vo. 23 pp. Fr. 0.75

——. Nociones y ejercicios de aritmética y geometría. Madrid, 1902. 8vo. 185 pp. Fr. 6.00

MARTINI ZUCCAGNI (A.). Trattato di algebra con i complementi di aritmetica razionale ad uso dei licei, secondo gli ultimi programmi governativi. Livorno, Giusti, 1903. 16mo. 12 + 328 pp. Fr. 2.60

MASSINGER (R.). See HOLZMANN (A.).

MOČNIK (F. VON). Lehrbuch der Arithmetik und Algebra nebst einer Aufgabensammlung für die oberen Klassen der Realschulen, bearbeitet von A. Neumann. 26ste, veränderte Auflage. Leipzig, Freytag, 1902. 8vo. 324 pp. Cloth. M. 3.80

MORENO REY (S.) y CERUELO Y OBISPO (J.). Elementos de matemáticas. Trigonometría. 6a edición. Madrid, 1902. 8vo. 63 pp. Fr. 1.50

NATH (M.). See BORK (H.).

NEUMANN (A.). See MOČNIK (F. VON).

NIEWENGLOWSKI (B.). Cours d'algèbre, à l'usage des élèves de la classe de mathématiques spéciales et des candidats à l'Ecole normale supérieure et à l'Ecole polytechnique. 5e édition. Paris, Colin, 1902. 8vo. 6 + 390 pp.

PIROZHEIKOV (M. V.). See SERRET (J. A.).

REBIÈRE (A.). Trigonometry, with exercises. Translated by N. de Georges. 3d edition. St. Petersburg, 1902. 8vo. 207 pp. (Russian.) M. 2.50

ROEDER (H.). Der Koordinatenbegriff und einige Grundeigenschaften der Kegelschnitte. Zunächst eine Ergänzung zur Neubearbeitung der Planimetrie von Kambly. Zum Gebrauche an Gymnasien bearbeitet. 2te, vermehrte Auflage. Breslau, Hirt, 1902. 8vo. 64 pp. Boards. M. 0.60

SALOMON (A.). Leçons d'arithmétique (compléments), à l'usage de l'enseignement secondaire des jeunes filles (classe de quatrième année). 2e édition. Paris, Nony, 1903. 18mo. 87 pp. Fr. 1.00

SCHRÖDER (T.). Beispiele und Aufgaben aus der Algebra, für Gymnasien, Realschulen und zum Selbstunterricht. Ergänzung zu W. Hechts und H. Lothers Buchstabenrechnung, als zweiter Teil der Aufgabensammlung aus der allgemeinen Zahlenlehre. 11te Auflage. Nürnberg, Korn, 1903. 8vo. 6 + 160 pp. Cloth. M. 1.80

SCHUSTER (M.). Geometrische Aufgaben und Lehrbuch der Geometrie. Planimetrie; Stereometrie; ebene und sphärische Trigonometrie. Nach konstruktiv-analytischer Methode bearbeitet. Ausgabe A: für Vollanstalten. Teil 2: Trigonometrie. Leipzig, Teubner, 1903. 8vo. 7 + 112 pp., 1 plate. Cloth. M. 1.60

SERRET (J. A.). Spherical trigonometry. Translated from the 8th French edition by M. V. Pirozheikov. St. Petersburg, 1902. 8vo. 86 pp. (Russian.) M. 2.00

SIEFERT (H. O. R.). Principles of arithmetic, embracing common fractions, percentage, proportion, involution, evolution, and mensuration; a manual for teachers and normal students. Boston, Heath, 1902. 16mo. 163 pp. Cloth. $0.75

SIEVERT (H.). Lehrbuch der Elementargeometrie zum Gebrauche an Mittelschulen und beim Selbstunterrichte. Teil I: Geometrie der Ebene. Abteilung 2: Harmonische Beziehungen, Pol und Polare, Potenz- und Aehnlichkeitsbeziehungen der Kreise. Leipzig, Deichert, 1902. 8vo. 3 + 37 pp. M. 0.90

——. Teil III: Geometrie des Raumes. Abteilung 2: Das Dreikant mit Kugeldreieck; sphärische Trigonometrie. Leipzig, Deichert, 1902. 8vo. 3 + 38 pp. M. 0.90

SHMULEVICH (P. K.). Collection of problems set at examinations for admission to higher technical schools. Part II: Algebra. 3d edition. St. Petersburg, 1902. 8vo. 313 pp. (Russian.) M. 6.00

SMITH (T.). Euclid, his life and system. London and New York, Scribner, 1902. 12mo. 4 + 227 pp. Cloth. (World's epoch makers.) $1.25

SUÁREZ SOMONTE (I.). Aritmética. Madrid, 1902. 8vo. 236 pp. Boards. Fr. 6.50

——. Algebra. Madrid, 1902. 8vo. 210 pp. Fr. 6.00

——. Programa de aritmética. Madrid, 1902. 8vo. 16 pp. (Instituto del Cardenal Cisneros.) Fr. 1.00

——. Programa de algebra y trigonometría. Madrid, 1902. 8vo. 24 pp. (Instituto del Cardenal Cisneros.) Fr. 1.00

SYMONDS (H. C.). Abstracts of the elements of arithmetic and elements of algebra arranged in tabular form. Revised edition. New York, Harison [1902]. 12mo. 8 + 102 pp. Cloth. (West Point series.) $0.80

TROPFKE (J.). Geschichte der Elementarmathematik in systematischer Darstellung. (In 2 volumes.) Vol. I: Rechnen und Algebra. Leipzig, Veit, 1902. 8vo. 8 + 332 pp. M. 8.00

TUCKEY (C. O.). Examples in algebra. London, Bell, 1902. 12mo. 186 pp. Cloth. 3s.

VACQUANT (C.). et MACÉ DE LÉPINAY (A.). Géométrie élémentaire. Nouvelle édition, conforme aux programmes du 31 mai 1902. Paris, Mason, 1903. 16mo. 352 pp. Fr. 3.00

WROBEL (E.). Uebungsbuch zur Arithmetik und Algebra. Teil I: Resultate. 2te Auflage. Rostock, Koch, 1902. 8vo. 92 pp. M. 2.80

ZIEGLER (F.). Graphische Darstellung der trigonometrischen Funktionen nebst Tafeln zur Konstruktion bestimmter Winkel und Linien. Praktisches Hilfsmittel beim geometrischen Zeichnen. Wiesbaden, 1902. 8vo. 22 pp., 6 plates. Cloth. M. 3.00

ZWERGER (M.). Leitfaden zum Unterrichte in der elementaren Mathematik, mit einer Sammlung von Aufgaben. 12te Auflage des Leitfadens der Mathematik von H. Müller. 3te Abteilung: Trigonometrie. München, Lindauer, 1902. 8vo. 7 + 55 pp. Cloth. M. 1.20

III. APPLIED MATHEMATICS.

ALESSANDRI (C.). Potenziale nei campi di forze newtoniane. Bergamo, Istituto italiano d'arti grafiche, 1902. 8vo. 29 pp. Fr. 1.00

BOULANGER (J.) et FERRIÉ (G.). La télégraphie sans fil et les ondes électriques. 4e édition. Paris, Berger-Levrault, 1902. 8vo. 190 pp. Fr. 3.00

BUCHHOLZ (H.). Untersuchung der Bewegung vom Typus $\frac{2}{3}$ im Problem der drei Körper und der "Hilda-Lücke" im System der kleinen Planeten auf Grund der Gyldenschen Störungstheorie. (Habilitationsschrift.) Halle, 1902. 4to. 165 pp.

BURZHEIEV (S.). Text-book of mechanics, with problems. 5th edition. St. Petersburg, 1902. 8vo. 275 pp. (Russian.) M. 5.00

CHARLIER (C. L.). Die Mechanik des Himmels. Vorlesungen. (In 2 volumes.) Vol. I. Leipzig, Veit, 1902. 8vo. 8 + 488 pp. M. 18.00

CHWOLSON (O. D.). Lehrbuch der Physik. Band I: Einleitung; Mechanik; einige Messinstrumente und Messmethoden; die Lehre von den Gasen, Flüssigkeiten und festen Körpern. Uebersetzt von H. Pflaum. Braunschweig, Vieweg, 1902. 8vo. 20 + 791 pp. M. 12.00

DIETSCH (C.). Leitfaden der darstellenden Geometrie; mit sehr vielen Aufgaben. 4te, nach der neuen Rechtschreibung verbesserte, sonst fast unveränderte Auflage. Erlangen, Deichert, 1903. 8vo. 4 + 152 pp. M. 2.20

EGGERS (W.). Lehrbuch der Schattenkonstruktion. Leipzig, Seemann, 1902. 4to. 6 + 42 pp., 21 plates. Cloth. M. 3.00

FRITSCH (H.). Die Huygenssche Darstellung des Lichtäthers, weitergeführt. Königsberg, 1902. 4to. 14 pp. M. 1.50

GRASSI (U.). Studi d'idrodinamica. Pisa, Nistri, 1902. 8vo. 84 pp.

GREINACHER (H.). Einführung in die Theorie der Doppelbrechung; elementar-geometrisch dargestellt. Eine Ergänzung zu den physikalischen Lehrbüchern. Leipzig, Veit, 1902. 8vo. 64 pp. M. 1.20

HOWE (M. A.). The design of simple rooftrusses in wood and steel, with an introduction to the elements of graphic statics. New York, Wiley, 1902. 8vo. 10 + 129 pp. Cloth. $2.00

LA COUR (J. L.). Theorie der Wechselströme und Transformatoren. Berlin, Springer, 1902. 8vo. 11 + 425 pp. Cloth. (Die Wechselstromtechnik, herausgegeben von E. Arnold, Vol. I.) M. 12.00

LAMBERT (J. H.). Abhandlungen zur Bahnbestimmung der Cometen: Insigniores orbitae cometarum proprietates (1761); Observations sur l'orbite apparente des comètes (1771); Auszüge aus den "Beiträgen zum Gebrauche der Mathematik" (1772). Deutsch herausgegeben und mit Anmerkungen versehen von J. Bauschinger. Leipzig, Engelmann, 1902. 12mo. 149 pp. Cloth. (Ostwald's Klassiker der exakten Wissenschaften, No. 133.) M. 2.40

LORENZ (H.). Lehrbuch der technischen Physik. (In 4 Bänden.) Band I: Technische Mechanik starrer Systeme. München, Oldenbourg, 1902. 8vo. 24 + 625 pp. M. 15.00

LUBELZA (J.). La acción centrífuga en la rotación de un sistema invariabile. Madrid, 1902. 8vo. 16 pp.

MICHELSON (A. A.). The velocity of light. Chicago, The University of Chicago Press, 1902. 4to. 12 pp. (Preprint from the University of Chicago decennial publications, Vol. 9.) $0.25

MÜLLER-BRESLAU (H. F. B.). Die graphische Statik der Baukonstruktionen. 3te, wesentlich vermehrte Auflage. Band II, Abteilung 1: Formänderung ebener Fachwerke; Untersuchung der ebenen, statisch unbestimmten Fachwerke. Leipzig, Baumgärtner, 1903. 8vo. 12 + 48 pp., 7 plates. M. 16.00

PEIRCE (B. O.). Elements of the theory of the Newtonian potential function. 3d revised and enlarged edition. Boston, Ginn, 1902. 12mo. 14 + 490 pp. Cloth. $2.50

PFLAUM (H.). See CHWOLSON (O. D.).

PULLEN (W. W. F.). Mechanics; theoretical, applied, and experimental. London, Longmans, 1902. 12mo. 390 pp. Cloth. 4s. 6d.

REISSNER (H.). Schwingungsaufgaben aus der Theorie des Fachwerks. (Diss.) Berlin, 1902. 8vo. 28 pp.

RÖDIGER (C.). Untersuchungen über das Doppelsternsystem Algol. (Diss.) Jena, 1902. 8vo. 34 pp.

STEINGRÄBER (E.). Ueber die Influenz, welche elektrische Masse auf ein hohles Rotationsellipsoid als Konductor ausübt, wenn die Masse im Innern auf der Achse symmetrisch angelagert ist. (Diss.) Greifswald, 1902. 8vo. 24 pp.

STEINMETZ (C. P.). Theoretical elements of electrical engineering. 2d edition, revised and corrected. New York, Electrical World and Engineer, 1902. 8vo. 7 + 330 pp. Cloth. $2.50

WISKOCZIL (E.). Unmittelbare Darstellung der einzelnen Bilder der regelmässigen Vielflächner. (Progr.) Iglau, 1902. 8vo. 14 pp., 3 plates.

WOOD (DE V.). The principles of elementary mechanics. Re-edited with emendations by A. J. Wood. 10th edition. New York, Wiley, 1902. 8vo. 16 + 347 pp. Cloth. $1.25

ZIMMERMANN (W.). Eine Methode zur Berechnung specieller Störungen durch Variation der kanonischen Elemente. (Diss.) Breslau, 1902. 4to. 30 pp., 3 plates.